Approximation of Functions by Certain Positive Linear Methods of Convergence

A Thesis
Submitted for the award of degree of
Doctor of Philosophy
in Mathematics
by

Km. Lipi
(2K18/PHD/AM/07)

Under the supervision of
Prof. Naokant Deo

Department of Applied Mathematics
Delhi Technological University
Bawana Road, Delhi 110042 (India)

APRIL, 2024

Dedicated to my Parents,

For their love and selfless sacrifices.

Certificate

Department of Applied Mathematics
Delhi Technological University, Delhi

This is to certify that the research work embodied in the thesis entitled "Approximation of Functions by Certain Positive Linear Methods of Convergence" submitted by Km. Lipi (2K18/PHD/AM/07) is the result of her original research carried out in the Department of Applied Mathematics, Delhi Technological University, Delhi, for the award of **Doctor of Philosophy** under the supervision of **Prof. Naokant Deo**.

It is further certified that this work is original and has not been submitted in part or fully to any other university or institute for the award of any degree or diploma.

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Date: APRIL, 2024
Place: Delhi , India

Prof. Naokant Deo
(Supervisor)

Prof. Ramesh Srivastava
(Head of Department)

Declaration

I declare that the research work in this thesis entitled "**Approximation of Functions by Certain Positive Linear Methods of Convergence**" for the award of the degree of ***Doctor of Philosophy in Mathematics*** has been carried out by me under the supervision of ***Prof. Naokant Deo***, Department of Applied Mathematics, Delhi Technological University, Delhi, India, and has not been submitted by me earlier in part or full to any other university or institute for the award of any degree or diploma.

I declare that this thesis represents my ideas in my own words and where others' ideas or words have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission.

Km. Lipi
(2K18/PHD/AM/07)

Date:

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Km. Lipi
DTU Delhi

Abstract

Approximation theory is indeed an old topic in mathematical analysis that continues to be an interesting field of research with several applications. After the well-known theorem due to Weierstrass and the important convergence theorem of Korovkin, many new operators were proposed and constructed by several researchers. The theory of these operators has been an important area of research in the last few decades. This thesis is mainly concerned with convergence estimates of several positive linear operators. The introductory chapter is a collection of relevant definitions and literature of concepts that are used throughout this thesis.

Several results have been established for different exponential-type operators, a notion that was first presented by May and then extensively investigated in cooperation with Ismail. The Bézier variant of these operators has been defined. Two decades ago it was observed that if we modify the original operators, we can have a better approximation. The basic properties and Voronoskaya type results for the approximation of exponential operators have been studied, and after being modified to preserve exponential functions, the results for improved error estimates have been achieved. A modification of certain Gamma type operators that preserves the test functions t^ϑ, $\vartheta = \{0\} \bigcup \mathbb{N}$ has been provided and rate of convergence for functions of bounded variation has been studied.
Some approximation properties of the Pólya distribution-based generalization of λ-Bernstein operators, such as rate of convergence, interpolation behavior, and the impact of changing parameter values, have been investigated. Certain theorems are derived to verify the convergence of generalized Bernstein operators based on shifted knots.

Some results have been proved for bivariate generalization of operators involving a class of orthogonal polynomials called Apostol-Genocchi polynomials. Furthermore, a conceptual extension for these bivariate operators, referred to as the "generalized boolean sum (GBS)", has been introduced with the goal of determining the degree of approximation for Bögel continuous functions.

Graphical illustration and tables that effectively showcase the convergence and demonstrate the approximation error have been included for all the operators.

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$\mathbb{N}$	the set of natural numbers
$\mathbb{N} \cup \{0\}$	the set of natural number including zero
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^{+}$	the set of positive real numbers
$[a, b]$	a closed interval
(a, b)	an open interval
Λ	index set
e_n	denotes the n-th monomials with $e_n : [a, b] \to \mathbb{R}$, $e_n(x) = x^n$, $n \in \mathbb{N}_0$
$(x)_n$	the rising factorial $(x)_n := x\,(x+1)\,(x+2)\ldots(x+n-1)\,,\ (x)_0 = 1$
$\Omega(f; \delta)$	the weighted modulus of continuity
$C[a, b]$	the set of all real-valued continuous function defined on compact interval $[a, b]$
$C^r[a, b]$	the set of all real-valued, r-times continuously differentiable function ($r \in \mathbb{N}$)
$C[0, \infty)$	the set of all continuous functions defined on $[0, \infty)$
$C_B[0, \infty)$	the set of all continuous and bounded functions on $[0, \infty)$
$C_B^r[0, \infty)$	the set of all r-times continuously differentiable functions in $C_B[0, \infty)$ ($r \in \mathbb{N}$)
$B_\rho[0, \infty)$	the set of all functions f defined on $[0, \infty)$ satisfying the condition : $\lvert f(x) \rvert \leq M\rho(x)$, M is a positive constant, and ρ is weight function.
$C_\rho[0, \infty)$	the subspace of all continuous function in $B_\rho[0, \infty)$

Chapter 1

Introduction

"Close enough is often close to perfect."

These words express how, in many cases, it is not necessary for something to be totally perfect; being near-perfect or close to your goal is sufficient. It acknowledges that striving for perfection can be time consuming and may not significantly improve the outcome, especially when a good approximation will do the job effectively. We can only make an effort to enhance the approximation and to minimize the errors introduced in the process. The concept of approximation dominates every area of research. Humankind has always sought to complete tasks as accurately as possible while minimising errors brought on by procedural, environmental, instrumental, or human factors.

In mathematics, the main focus of theory of approximation is on identifying the best ways to approximate functions with simpler ones and quantifying the errors that are introduced thereby. The foundation of approximation theory was laid on a result first given by Karl Weierstrass [170] in 1885, which states that for every continuous function f on a closed interval $[a,b]$ and any $\epsilon > 0$, there exists a polynomial p of degree n on $[a,b]$ such that

$$|f(x) - p(x)| < \epsilon, \ \forall x \in [a,b].$$

In other words, any continuous function on a closed and bounded interval can be uniformly approximated on that interval by polynomials to any degree of accuracy.

1.1 Preliminaries

In this section, we recall some definitions and properties regarding approximation operators discussed here that will be of interest to the whole thesis.

1.1.1 Positive Linear Operators

Definition 1.1.1 *Let X, Y be two linear spaces of real functions. Then the mapping L : $X \to Y$ is a linear operator if:*

$$L(\alpha f + \beta g; x) = \alpha L(f; x) + \beta L(g; x),$$

for all $f, g \in X$ and $\alpha, \beta \in \mathbb{R}$. If for all $f \in X$ and $f \geq 0$, it follows that $L(f; x) \geq 0$, then L is called a positive operator.

Next, we define the modulus of continuity, mainly used to measure quantitatively the uniform continuity of functions.

1.1.2 Usual and Higher Order Modulus of Continuity

Definition 1.1.2 *Let $f \in C[a, b]$ and $\delta \geq 0$, then*

$$\omega(f; \delta) = \sup\{|f(x+h) - f(x)| : x, x+h \in [a, b], 0 \leqslant h \leqslant \delta\}.$$

Here ω is known as the usual modulus of continuity or simply first order modulus of continuity which was introduced by Lebesgue in 1910.

Some of the error estimates in this thesis are given in terms of the modulus of continuity of higher order. Therefore we now give the definition of ω_r, $r \in \mathbb{N}$, as given in 1981 by Schumaker [148].

Definition 1.1.3 *Let $f \in C[a, b]$, then for $r \in \mathbb{N}$ and $\delta \geq 0$, the modulus of continuity of order r is defined as:*

$$\omega_r(f; \delta) = \sup\left\{\left|\Delta_h^r f(x)\right| : x, x+rh \in [a, b], 0 \leqslant h \leqslant \delta\right\}, \tag{1.1}$$

where

$$\Delta_h^r f(x) = \sum_{i=0}^{r} (-1)^{r-i} \begin{pmatrix} r \\ i \end{pmatrix} f(x+ih)$$

denotes the forward difference with step size h. In particular, for $r = 1$, $\omega(f; \delta)$ is the usual modulus of continuity.

If f is continuous and bouned function on $\mathbb{R}$, (1.1) will also define $\omega_r(f; \delta)$ for such a function.

Proposition 1.1.4 *The modulus of continuity of order r verifies the following properties:*

1. $\omega_r(f;\delta)$ *is a positive, monotonically increasing function on* $(0,\infty)$.
2. f *is uniformly continuous* $\Leftrightarrow \lim\limits_{\delta\to 0}\omega(f;\delta)=0$.
3. $\omega_r(f;nx)\leq n^r\omega_r(f;x)$ *for all* $n\in\mathbb{N}$.
4. $\omega_r(f;\lambda\delta)\leqslant(1+\lambda)\,\omega_r(f;\delta)$, *for any* $\lambda>0$.
5. $\omega_{r+1}(f;\delta)\leq 2\omega_r(f;\delta)$.

For $r=1$, *these properties are valid for the usual modulus of continuity* $\omega(f;.)$.

1.1.3 Peetre K-Functional

Definition 1.1.5 *For* $f\in C_B[a,b]$, *let us consider the following K-Functional,*

$$K_2(f;\delta)=\inf_{g\in C_B^2[a,b]}\left\{\left\|f-g\right\|+\delta\left\|g''\right\|,\delta>0\right\},$$

where $C_B^2[a,b]=\{g\in C_B[a,b]:g',g''\in C_B[a,b]\}$. From [74], there exists an absolute constant $C>0$, such that

$$K_2(f;\delta)\leqslant C\omega_2\left(f;\sqrt{\delta}\right). \tag{1.2}$$

1.1.4 Ditzian-Totik Modulus of Smoothness

We recall the definitions of the Ditzian-Totik first order modulus of smoothness and the K-functional [74]. Let $\varphi(x)=\sqrt{x(1-x)}$ and $f\in C[0,1]$, then the first order modulus of smoothness is defined as:

$$\omega_\varphi(f;\delta)=\sup_{0\leq h\leq\delta}\left\{\left|f\left(x+\frac{h\varphi(x)}{2}\right)\right|-\left|f\left(x-\frac{h\varphi(x)}{2}\right)\right|,x\pm\frac{h\varphi(x)}{2}\in[0,1]\right\}.$$

Further, the corresponding Peetre's K-functional is given by

$$K_\varphi(f;\delta)=\inf_{g\in W_\varphi[0,1]}\{||f-g||+\delta\,||\varphi g'||\},\quad\delta>0$$

where

$$W_\varphi[0,1]=\{g:||\varphi g'||<\infty,\ g\in AC_{loc}[0,1]\},$$

$AC_{loc}[0,1]$ denotes the space of all absolutely continuous functions on every interval $[a,b]\subset(0,1)$ and $||.||$ is the uniform norm in $C[0,1]$. Moreover, from [[74], p. 11], there exists a constant $C>0$ such that:

$$K_\varphi(f;\delta)\leq C\omega_\varphi(f;\delta).$$

1.1.5 Weighted Spaces and Corresponding Modulus of Continuity

Let $B_\rho(I)$ be the space of all functions f defined on the interval $I \in \mathbb{R}$ for which there exist a constant $C > 0$ such that $|f(x)| \leq C\rho(x)$, for every $x \in I$, where ρ is a positive continuous function called weight. In 1974, Gadjiev [80; 81] introduced the weighted space $C_\rho(I)$, which is the set of all continuous functions f on the interval $I \in \mathbb{R}$ and $f \in B_\rho(I)$. This space is a Banach space, endowed with the norm

$$\|f\|_\rho = \sup_{x\in I} \frac{|f(x)|}{\rho(x)}.$$

For $I = [0, \infty)$, the subspace $C^*_\rho[0, \infty)$ is defined as follows:

$$C^*_\rho[0, \infty) := \{f \in C_\rho[0, \infty) : \lim_{x\to\infty} \frac{|f(x)|}{\rho(x)} = k < +\infty\}.$$

Many authors [9; 100] use the following weighted modulus of continuity $\Omega(f;\delta)$ for $f \in C^*_\rho[0, \infty)$:

$$\Omega(f;\delta) = \sup_{x\in[0,\infty),|h|<\delta} \frac{|f(x+h) - f(x)|}{(1+h^2)(1+x^2)}.$$

Let us denote by $C^*[0, \infty)$, the Banach space of all real valued continuous functions on $[0, \infty)$ with the property that $\lim_{x\to\infty} f(x)$ exists and is finite endowed with the uniform norm. In [44], the following theorem is proved:

Theorem 1.1.6 *If the sequence $A_n : C^*[0, \infty) \to C^*[0, \infty)$ of positive linear operators satisfies the conditions*

$$\lim_{n\to\infty} A_n\left(e^{-kt}; x\right) = e^{-kx},\ k = 0, 1, 2$$

uniformly in $[0, \infty)$, then

$$\lim_{n\to\infty} A_n(f; x) = f(x),$$

uniformly in $[0, \infty)$, for every $f \in C^*[0, \infty)$.

1.1.6 Modulus of Continuity for Exponential Functions

To find rate of convergence of operators satisfying the conditions from the above theorem , we use the following modulus of continuity:

$$\omega^*(f;\delta) = \sup\left\{|f(x) - f(t)| : \ x, t \geqslant 0, \left|e^{-x} - e^{-t}\right| \leqslant \delta\right\}$$

defined for every $\delta \geq 0$ and every function $f \in C^*[0, \infty)$.

Proposition 1.1.7 *The modulus of continuity defined for exponential functions has the following properties:*

1. *$\omega^*(f;\delta)$ can be expressed in terms of usual modulus of continuity, by the relation*

$$\omega^*(f;\delta) = \omega(f^*;\delta)$$

where f^* is the continuous function on $[0,\infty)$ given by:

$$f^*(x) = \begin{cases} f(-\ln(x)), & x \in (0,\infty] \\ \lim_{t\to\infty} f(t), & x = 0 \end{cases}.$$

2. *For every $t, x \in [0,1]$ and $M > 0$, we have*

$$\omega^*(f;\delta) \leq \left(1 + e^M\right) \cdot \omega(f;\delta).$$

3. *The defined modulus of continuity ω^* possess the following property:*

$$|f(t) - f(x)| \leq \left(1 + \frac{(e^{-t} - e^{-x})^2}{\delta^2}\right)\omega(f^*;\delta).$$

1.1.7 Lipschitz Class

Definition 1.1.8 *A function $f(x)$ defined on an intercept $\langle a,b\rangle$ (this may mean the segment $[a,b]$ and the interval (a,b) which, specifically, may also be $(-\infty,+\infty)$, as well as one of the half segment, $[a,b)$ or $(a,b]$), and satisfying the condition*

$$|f(y) - f(x)| \leqslant M|y - x|^\beta \qquad (0 < \beta \leq 1),$$

for all pair of values x, y of this intercept is said to satisfy a Lipschitz condition. We write it as $f(x) \in Lip_M(\beta)$, the class of all those functions satisfying the Lipschitz condition with the exponent β and the coefficient M.

The definitions we provided above are for functions with a single variable. These definitions are slightly different for a function with two independent variables. In this thesis, we have also studied the bivariate generalization of some operators. Therefore we provide some definitions which will be used accordingly.

1.1.8 Total and Partial Modulus of Continuity

To establish the degree of approximation of bivariate operators in the space of continuous functions on compact set $I_{ab} = [0,a] \times [0,b]$, the total modulus of continuity for function $f \in C(I_{ab})$ is defined by:

$$\omega_{total}(f;\delta_1,\delta_2) = \sup\{|f(t_1,t_2) - f(x,y)| : (t_1,t_2),(x,y) \in I_{ab}, |t_1 - x| \leq \delta_1, |t_2 - y| \leq \delta_2\}.$$

Further, the partial moduli of continuity with respect to the independent variables x and y is given as:

$$\omega^1(f;\delta) = \sup\{|f(x_1,y) - f(x_2,y)| : 0 \leq y \leq b, |x_1 - x_2| \leq \delta\}$$

and

$$\omega^2(f;\delta) = \sup\{|f(x,y_1) - f(x,y_2)| : 0 \leq x \leq a, |y_1 - y_2| \leq \delta\}.$$

Both total and partial modulus of continuity for bivariate functions satisfy the properties of usual modulus of continuity and can be studied more in [26].

Next, to establish the rate of convergence of bivariate operators, we define the following Peetre's K-functional.

1.1.9 Peetre's K-Functional for Bivariate Operators

Let $C^2(I_{ab})$ denote the set of all continuous functions on $I_{ab} = [0,a] \times [0,b]$, whose first and second derivatives exist and are continuous on the interval I_{ab}. We define the norm:

$$\|f\|_{C^2(I_{ab})} = \|f\|_{C(I_{ab})} + \sum_{k=1}^{2}\left(\left\|\frac{\partial^k f}{\partial x^k}\right\|_{C(I_{ab})} + \left\|\frac{\partial^k f}{\partial y^k}\right\|_{C(I_{ab})}\right).$$

From Butzer and Berens[46], the Peetre's K-functional for $f \in C^2(I_{ab})$ is defined as:

$$K(f;\sigma) = \inf_{t \in C^2(I_{ab})}\left\{\|f - t\|_{C^2(I_{ab})} + \sigma\|t\|_{C^2(I_{ab})}, \sigma > 0\right\}.$$

For a positive constant M, Peetre's K-functional satisfies the following inequality:

$$K(f;\sigma) \leqslant M\left\{\varpi_2\left(f;\sqrt{\sigma}\right) + \min\{1,\sigma\}\|f\|_{C(I_{ab})}\right\}$$

where $\varpi_2\left(f;\sqrt{\sigma}\right)$ is the second order modulus of continuity for bivariate functions, defined as:

$$\varpi_2\left(f;\sqrt{\sigma}\right) = \sup\left\{\left|\sum_{i=0}^{2}(-1)^{2-i} f(x + ix_0, y + iy_0)\right| \right.$$
$$\left. : (x,y), (x + ix_0, y + iy_0) \in I_{ab}, |x_0| \leqslant \sigma, |y_0| \leqslant \sigma\right\}.$$

1.2 Historical Background and Literature Review

A simple yet powerful tool for deciding whether a given sequence of positive linear operators on $C[0,1]$ or $C[0,2\pi]$ is an approximation process or not are the Korovkin theorems. These theorems are an abstract results in approximation which gives conditions for uniform approximation of continuous functions on a compact metric space. The Korovkin theorem [[25] pp.218] elegantly says that if $(L_n)_{n\geq 1}$ is an arbitrary sequence of positive linear operators on the space $C[a,b]$, and if

$$\lim_{n\to\infty} L_n\left(e_i;x\right) \to e_i \text{ uniformly on } [a,b],$$

for the test functions $e_i(x) = x^i$, $i = 0,1,2$ then

$$\lim_{n\to\infty} L_n\left(f;x\right) \to f \text{ uniformly on } [a,b],$$

for each $f \in C[a,b]$.

The above theorem, known as Korovkin's first theorem, was proposed by Korovkin [119] in 1953. Korovkin's second theorem has a similar statement, but the space $C[0,1]$ is replaced by the space $C[0,2\pi]$, i.e. the space of all 2π periodic real-valued functions on $\mathbb{R}$. The test functions e_i in this case belong to the set $\{1,\cos(x),\sin(x)\}$ for $i = 0,1,2$ respectively. H. Bohmann [43] in 1952 had proved a result similar to Korovkin's first theorem but concerning sequences of positive linear operators on $C[0,1]$ of the form

$$L_n\left(f;x\right) = \sum_{i\in I} f\left(a_i\right)\phi_i, \quad f \in C[0,1]$$

where $(a_i)_{i\in\Lambda}$ is a finite set of numbers in $[0,1]$ and $\phi_i \in C[0,1]$, $i \in \Lambda$. Therefore, Korovkin's first theorem is also known as Bohman-Korovkin Theorem. An immediate analogue of Korovkin's theorem does not hold if the domain of definition of the function f becomes unbounded and hence requires the function to have some finite limit at infinity. For continuous and unbounded functions on $[0,\infty)$, Gadžiev [80] in 1974 introduced a weighted space $C_\rho[0,\infty)$ defined as the set of all continuous functions f on the interval $[0,\infty)$ for which there exists a positive constant M such that $|f(x)| \leq M\rho(x)$, for every $x \in [0,\infty)$. Here ρ is a positive continuous function called the weight function. The space $C_\rho[0,\infty)$ is a Banach space equipped with the norm

$$\|f\|_\rho = \sup_{x\in[0,\infty)} \frac{|f\left(x\right)|}{\rho\left(x\right)}.$$

The Korovkin theorem by Gadžiev is given as: Let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous, strictly increasing and unbounded function. Set $\rho\left(x\right) = 1 + \varphi^2\left(x\right)$. If the sequence of positive linear operators $L_n : C_\rho\left[0,\infty\right) \to C_\rho\left[0,\infty\right)$ verifies

$$\lim_{n\to\infty} \left\|L_n(\varphi^i;x) - \varphi^i(x)\right\|_\rho = 0, \ i = 0,1,2$$

Then,

$$\lim_{n\to\infty} \|L_n(f;x) - f(x)\|_\rho = 0$$

for every $f \in C_\rho\left[0,\infty\right)$ for which $\lim\limits_{n\to\infty} \frac{f(x)}{\rho(x)}$ exists and is finite.

With the application of Korovkin theorems to study the uniform convergence of positive linear operators, advancement in approximation theory began with the development of new positive linear operators, the first and most important of which are the Bernstein polynomials. In 1912, Bernstein [39] gave an elegant proof of the famous Weierstrass approximation theorem by defining a sequence of polynomials called Bernstein operators on the closed interval $[0, 1]$ (extended on $[a, b]$ by simple manipulations). These operators are defined as:

Let f be a bounded function on $[0, 1]$. The Bernstein operators of degree n with respect to f is defined as:

$$B_n(f; x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0, 1] \tag{1.3}$$

where

$$b_{n,k}(x) = \begin{pmatrix} n \\ k \end{pmatrix} x^k (1-x)^{n-k}, k = 0, 1, 2, \cdots, n$$

and $\begin{pmatrix} n \\ k \end{pmatrix} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}$ represents the binomial coefficient. It should be noted that $b_{n,k}(x) \in P_n,\ k = 0, 1, 2, \cdots, n$ where P_n denotes the space of all polynomials of degree at most n, are the so-called Bernstein polynomials.

Since the Bernstein operators were only suitable for approximating functions on a compact interval, Szász in 1950 [163], and Mirakjan in 1941 presented a generalization of these operators for a continuous function f on the interval $[0, \infty)$ which later came to be known as Szász-Mirakjan operators. These operators are defined as:

$$S_n(f; x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0, \infty), \tag{1.4}$$

where

$$s_{n,k}(x) = e^{-nx} \frac{(nx)^k}{k!}.$$

In 1957, Baskakov [36] introduced another sequence of positive linear operators on the interval $[0, \infty)$ called Baskakov operators which are defined as:

$$V_n(f; x) = \sum_{k=0}^{\infty} v_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0, \infty), \tag{1.5}$$

where

$$v_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}.$$

To approximate integrable functions on the compact interval $[a, b]$, Kantorovich [114] was the first to define the integral variant of Bernstein operators by replacing the weight function with the average mean of the weight function in the vicinity of the point $\frac{k}{n}$ as:

$$\hat{B}_n(f; x) = (n+1) \sum_{k=0}^{n} b_{n,k}(x) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt.$$

where $b_{n,k}(x)$ is defined in (1.3). Similarly, Szász Kantorovich operators on the unbounded interval $[0, \infty)$ for given basis function $s_{n,k}(x)$ in (1.4) are defined as:

$$\hat{S}_n(f; x) = n \sum_{k=0}^{\infty} s_{n,k}(x) \int_{k/n}^{(k+1)/n} f(t)\, dt. \tag{1.6}$$

For Baskakov operators, the integral variant on the semi real axis is:

$$\hat{V}_n(f; x) = (n-1) \sum_{k=0}^{\infty} v_{n,k}(x) \int_{k/(n-1)}^{(k+1)/(n-1)} f(t)\, dt. \tag{1.7}$$

where $v_{n,k}(x)$ is defined in similar manner as in (1.5). To estimate functions on an unbounded interval, Kantorovich forms of various approximation operators have been defined from time to time. For further reference, one can visit the articles [8; 15; 24; 59; 84; 134].

In 1967, Durrmeyer [77] gave a more generalized integral modification of Bernstein operators by replacing the values of $f(k/n)$ by an integral over the weight function on the interval $[0, 1]$. These so called Bernstein-Durrmeyer operators were first studied by Derrienic [66] and are defined as:

$$\tilde{B}(f; x) = (n+1) \sum_{k=0}^{n} b_{n,k}(x) \int_{0}^{1} b_{n,k}(t) f(t)\, dt. \tag{1.8}$$

In 1985, Mazhar and Totik [131] introduced the Szász-Durrmeyer operators as follows:

$$\tilde{S}_n(f; x) = n \sum_{k=0}^{\infty} s_{n,k}(x) \int_{0}^{\infty} s_{n,k}(t) f(t)\, dt. \tag{1.9}$$

In the same year Sahai and Prasad [161] also established the Baskakov-Durrmeyer operators defined as follows:

$$\tilde{V}_n(f; x) = (n-1) \sum_{k=0}^{\infty} v_{n,k}(x) \int_{0}^{\infty} v_{n,k}(t) f(t)\, dt. \tag{1.10}$$

where $b_{n,k}(x)$, $s_{n,k}(x)$ and $v_{n,k}(x)$ are same as in (1.3), (1.4) and (1.5) respectively. Durrmeyer type variants of a number of positive linear operators were constructed in subsequent years. One can refer to the articles [4; 18; 57; 83; 111].

Bernstein polynomials, with their helpful structure and applications in various areas (computer technologies, engineering sciences, physics, etc.), have been the subject of intense research for more than a century. A variety of modifications and generalizations of Bernstein polynomials have also been investigated in the literature. One of the main goals of these modifications and generalizations are to transfer Bernstein polynomials over unbounded intervals in order to extend the class to which the desired function belongs. For example, Chlodowsky [56], transferred these polynomials from $[0, 1]$ to $[0, b_n]$ $\left(b_n \to \infty, \frac{b_n}{n} \to 0\right)$ by introducing a new version of Bernstein polynomials. Usta [167] proposed a new modification of Bernstein operators which fix constant and preserve Korovkin's other test functions in limit case. Another goal of continuing research on Bernstein polynomials is to improve approximation speed and reduce the absolute error that occur as a natural outcome of the approximation process. Gadjiev and Ghorbanalizadeh [82] carried out one of these research and defined Bernstein-Stancu type polynomials with shifted knots. Shifted knots have the benefit of allowing approximation on interval $(0, 1)$ and its subinterval. It also increases the flexibility of operators for approximation.

Depending on the parameter λ, Cai et al. [50] proposed and took into consideration a new generalization of Bernstein polynomials known as λ-Bernstein operators. When $\lambda = 0$, these λ-Bernstein operators reduce into the well-known Bernstein operators [39]. In this thesis, the generalization of λ-Bernstein operators [50] based on Pólya distribution is presented.

Zeng and Piriou [174] pioneered the study of Bézier variant of Bernstein operators. It is well known that Bézier curves are the parametric curves used in computer graphics and designs, interpolation, approximation, curve fitting etc. In graphics of vectors, these are used to model smooth curves and also used in animation designs. These curves were invented by Pierre Etienne Bézier, a French engineer at Renault. Later on, Chang [54] introduced Bézier variant for generalized Bernstein operators and studied some of its approximation properties. Zeng and Chen [175] introduced the Bézier Bernstein-Durrmeyer operators and studied the rate of convergence for these operators. These works have been continued ever since by several authors with construction of Bézier variants of different operators and analyzed their approximation properties. For further references, one can refer to articles [23; 67; 91; 96; 97; 104; 144; 153; 157; 158].

The operators we have covered up to this point were only appropriate for approximating functions with one variable. So Kingsley [176] initiated the study of Bernstein operators for the two variable case for the class of k-times continuously differentiable functions on a closed and bounded rectangle region. Butzer [177] investigated some approximation properties for these operators. After that, Stancu [178] introduced another kind of generalization of Bernstein operators for the two and several variable case. In last couple of years, researchers have proposed bivariate of various operators and demonstrated their convergence behaviour. Readers can go through these papers [19; 34; 61; 89; 103; 123; 134; 140] to get some knowledge about this topic.

In the past few decades, GBS operators have gained substantial importance amongst researchers to study functions in Bögel space. In 1934, Bögel [41; 42] developed the study of B-continuous and B-differentiable functions. Later, Dobrescu and Matei [75] illustrated that the convergence of GBS operators corresponding to the bivariate Bernstein polynomial is uniform. Badea and Cottin [33], instituted Korovkin-type theorem for GBS operators. Thereafter Badea et al. [31] proved the well known "Test function theorem" to approximate these kinds of functions. Badea and Badea [32] for these functions established a quantitative variant of the Korovkin-type theorem. In recent years many researchers [13; 38; 93; 98] contributed in the area of approximation theory.

With the advancement in approximation theory, researchers were drawn to developing novel approximation operators that had faster convergence rates and were applicable within a variety of functions and spaces. May [129] in 1976 first defined operators of the form:

$$W_\lambda(f;x)=\int_{-\infty}^{\infty}S(\lambda,x,t)\,f(t)\,dt,$$

and termed it as exponential operators provided they satisfy two conditions, first is the homogenous partial differential equation

$$\frac{\partial}{\partial x}S(\lambda,x,t)=\frac{\lambda(t-x)}{q(x)}S(\lambda,x,t), \tag{1.11}$$

where, $S(\lambda,x,t)\geq 0$ is the kernel of these operators and q is a polynomial of at most degree n which is analytic and positive for $x\in(a,b)$ for some a,b such that $-\infty\leq a\leq b\leq+\infty$, while second is the normalization condition

$$W_\lambda(1;x)=\int_{-\infty}^{\infty}S(\lambda,x,t)dt=1. \tag{1.12}$$

Operators satisfying the above conditions are, for example, the Bernstein operators, Szász Mirakjan operators, Post-Widder operators, Gauss-Weierstrass operators and Baskakov operators. These well-known operators are thus referred to as exponential operators. Some approximation properties were also studied for polynomials of degree at most 2.

A year later, Ismail and May [102] proved that for a polynomial q of any degree, the approximation operators W_λ can be uniquely determined and satisfies the differential equation (1.11) along with the normalisation condition (1.12). As a consequence of this, they recovered some known operators for constant, linear and quadratic polynomials. Further, they gave some new operators for cubic polynomials q such as: for the polynomial $q(x) = x(1+x)^2$, new exponential operators derived by Ismail and May using the method of bilateral Laplace transform are defined as:

$$R_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{n(n+k)^{k-1}}{k!} (xe^{-x})^k f\left(\frac{k}{n+k}\right), \quad n \in \mathbb{N} \;\; x \in (0,1). \tag{1.13}$$

The convergence properties and the corresponding Kantorovich variant of these operators were extensively studied in [134]. A complete asymptotic expansion for this sequence of operators is also derived in [2]. Another exponential operators corresponding to $q(x) = 2x^{3/2}$ is also studied in [1; 122]. Sato [162] studied the global behaviour of exponential operators like Bernstein, Szász Mirakjan, Gauss-Weierstrass etc., in weighted spaces. Totik [165] described their theoretical approximation properties from the point of view of global uniform approximation. For more information on exponential operators, one can refer to the book [94] and the articles cited therein. In this thesis, we have investigated some modifications of exponential operators given by Ismail and May [102] and discussed their convergence properties.

Another approximation operators examined in this thesis are the Gamma-type operators. A vital tool among the researchers to study positive linear operators is Euler's gamma function, which for $r > 0$ is defined as follows:

$$\Gamma(r) = \int_0^{\infty} e^{-t} t^{r-1} dt.$$

For $(a, b) \in \mathbb{R}$, Miheşan [132] defined a more general linear transform of f, also called the $(a, b)-$gamma transform, as follows:

$$\Gamma(r)^{(a,b)}(f;x) = \frac{1}{\Gamma(r)} \int_0^{\infty} e^{-t} t^{r-1} f\left(xe^{-bt}\left(\frac{t}{r}\right)^a\right) dt. \tag{1.14}$$

The transform (1.14) reproduces distinct integral operators for different values of a, b and r. The derived operators have been introduced and studied extensively by researchers over the past few decades; for instance, see [28; 95; 121; 126; 138]. For $b = 0$, $a = -1$ and $r = n + 1$, one can obtain particular operators first introduced and studied by Lupaş and Müller [127] and also referred to as Gamma operators. In this thesis work, we have proposed certain Gamma-type operators which possess the property of reproducing polynomial functions of the form $t^{\vartheta}, \vartheta \in \mathbb{N}$.

1.3 Improvement in Order of Approximation

The central idea in approximation theory is to estimate the rate of convergence of the sequence of operators using various convergence methods. These methods aim to improve the rate of convergence of operators, thereby reducing the error induced during the approximation process.

Our interest in this thesis is to improve the order of approximation of classical and existing operators. An important approach to improve the order of approximation was given by King in his pioneer work [117]. He presented a non-trivial sequence of positive linear operators defined on $C[0,1]$ that preserved the test functions e_0 and e_2. Let $\{r_n(x)\}$ be a sequence of continuous functions defined on $[0,1]$ such that $r_n(x) \in [0,1]$. Then the operators $V_{n,r_n} : C[0,1] \to C[0,1]$ are defined as:

$$V_{n,r_n}(f;x) = \sum_{k=0}^{n} (r_n(x))^k (1 - r_n(x))^{n-k} f\left(\frac{k}{n}\right), \quad x \in [0,1]$$

where

$$r_n(x) = \begin{cases} x^2, & n = 1 \\ \frac{-1}{2(n-1)} + \sqrt{\left(\frac{n}{n-1}\right)x^2 + \frac{1}{4(n-1)^2}}, & n = 2,3,\cdots \end{cases}$$

The operators V_{n,r_n} interpolate f at the endpoints 0 and 1 and are not polynomial operators. King also proved that the order of approximation of operators V_{n,r_n} is at least as good as the order of approximation of Bernstein operators for $x \in [0, \frac{1}{3}]$. Inspired by his work, other modifications of well-known operators were constructed as well to fix certain functions and to study their approximation and shape-preserving properties. In [52] Cárdenas-Morales et al. presented a family of sequences of linear Bernstein-type operators $B_{n,\alpha}$, $n > 1$, depending on a real parameter $\alpha \geq 0$, and fixing the polynomial function $e_2 + \alpha e_1$. Among other things, the authors prove that if f is convex and increasing on $[0,1]$, then $f(x) \leq B_{n,\alpha}(f;x) < B_n(f;x)$ for every $x \in [0,1]$. Duman and Özarsalan [76] gave a modification of the classical Szász-Mirakjan operators to provide a better error estimation. Ozsarac and Acar [141] presented a new modification of the Baskakov operators, which preserve the functions $e^{\mu t}$ and $e^{2\mu t}, \mu > 0$.

In this thesis, we have used King's approach to present a better modification of various operators, thereby reducing the error and improving the rate of approximation of the considered operators.

1.4 Chapter-wise Overview of the Thesis

The thesis consists of six chapters, whose contents are described below:

Chapter 1 provides an in-depth analysis of the literature and historical context of some key approximation operators. Along with a brief summary of the chapters this thesis is divided into, we also discuss some preliminary instruments that will be employed subsequently to derive our main results.

Chapter 2 is dedicated to some exponential operators, a concept first studied by May and later explored thoroughly in collaboration with Ismail. This chapter is majorly divided into two distinct sections. The initial section considers an exponential operators associated with the polynomial $x(1+x)^2$ which is defined as:

$$\tilde{R}_\lambda(f;t) = e^{-\frac{\lambda t}{(1+t)}} \sum_{k=0}^{\infty} \frac{\lambda(\lambda+k)^{k-1}}{k!} \left(\frac{t}{1+t}\right)^k e^{-\frac{\lambda t}{(1+t)}} f\left(\frac{k}{\lambda}\right), \quad \lambda > 0, t \in (0,\infty).$$

With change of variables $x = \frac{t}{1+t}$, Ismail and May [102] defined positive linear operators for a continuous function $f \in C[0,1]$, as follows:

$$R_\lambda(f;t) = e^{-\lambda x} \sum_{k=0}^{\infty} \frac{\lambda(\lambda+k)^{k-1}}{k!} (xe^{-x})^k f\left(\frac{k}{\lambda+k}\right).$$

We studied the approximation properties of Ismail-May operators [102] based on a non-negative real parameter λ. We provide some graphs and an error estimation table for a numerical example depicting the convergence of our proposed operators. We further define the Bézier variant of these operators in the following way:

$$R_\lambda^\alpha(f;t) = \sum_{k=0}^{\infty} f\left(\frac{k}{\lambda+k}\right) \vartheta_{n,k}^\alpha(x), \quad \alpha \geq 1, \ x \in [0,1],$$

where $\vartheta_{\lambda,k}^\alpha(x) = (P_{\lambda,k}(x))^\alpha - (P_{\lambda,k+1}(x))^\alpha$, and $P_{\lambda,k}(x) = \sum_{j=k}^{\infty} r_{\lambda,j}(x)$ for $k = 0,1,2,..\lambda$ are the Bézier basis functions and

$$r_{\lambda,k}(x) = e^{-\lambda x} \frac{\lambda(\lambda+k)^{k-1}}{k!} (xe^{-x})^k, \quad x \in [0,1].$$

We established a direct approximation theorem using the Ditizan-Totik modulus of smoothness and a Voronovoskaya-type asymptotic theorem. We also study the error in the approximation of functions having derivatives of bounded variation. Lastly, we present the bivariate generalization of Ismail May operators as follows:

$$R_{\lambda_1\lambda_2}^{k_1k_2}(f;x,y) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \hbar_{\lambda_1\lambda_2}^{k_1k_2}(x,y) f\left(\frac{k_1}{\lambda_1+k_1}, \frac{k_2}{\lambda_2+k_2}\right), \tag{1.15}$$

where the basis function is considered as:

$$\hbar_{\lambda_1\lambda_2}^{k_1k_2}(x,y) = e^{-(\lambda_1 x + \lambda_2 y)} \frac{\lambda_1\lambda_2(\lambda_1+k_1)^{k_1}(\lambda_2+k_2)^{k_2}}{k_1!k_2!} (xe^{-x})^{k_1} (ye^{-y})^{k_2}.$$

We estimate its rate of convergence for functions of the Lipschitz class.

The subsequent section is devoted to a modification of other exponential operators proposed by Ismail and May [102] associated with the polynomial $2x^{3/2}$ and is defined as:

$$\ddot{\mathcal{T}}_n(f;x) = e^{-n\sqrt{\varrho_n(x)}}\left\{ f(0) + n\int_0^\infty e^{-nt/\sqrt{\varrho_n(x)}} t^{-1/2} I_1\left(2n\sqrt{t}\right) f(t)\,dt \right\}.$$

The computation of the function $\varrho_n(x) = \frac{x^{3/2}\sqrt{4n^2+x+2n^2x+x^2}}{2n^2}$ is based on the assumption that these operators preserve the exponential functions of the form e^{-x}. The moments of the modified operators are achieved by using the concept of moment-generating function, with the aid of Mathematica software. This study aims to demonstrate the uniform convergence of these modified operators and analyze their asymptotic behavior through the Voronovskaya-type theorem. Furthermore, we confirm our claim by presenting graphical evidence that our modified operators possess better approximation in comparison to the original operators for a certain family of functions. Finally, the graphs for the convergence of the modified operators is achieved by employing Mathematica software.

Over the past few decades, scholars have dedicated their efforts to investigating a wide range of approximation operators in light of the advancements made in the theory of the gamma function. **Chapter 3** therefore focuses mainly on investigating a modification of certain Gamma-type operators. In this study, we used King's approach [117] to present a modification of certain Gamma-type operators. The objective of this modification is to ensure the preservation of the test functions $t^{\vartheta}, \vartheta \in \mathbb{N}$. The modified operators are defined as:

$$\mathbb{G}_n^{(\vartheta)}(f;x) = \frac{(2n+3)!\left(\beta_n^{(\vartheta)}(x)\right)^{n+3}}{n!\,(n+2)!}\int_0^\infty \frac{t^n}{\left(\beta_n^{(\vartheta)}(x)+t\right)^{2n+4}} f(t)dt,$$

where

$$\beta_n^{(\vartheta)}(x) = \left(\frac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\right)^{1/\vartheta} x.$$

The notation $(n)_\vartheta$ is used to represent the rising factorial, where $(n)_\vartheta = n(n+1)\ldots(n+\vartheta-1)$ and $(n)_0 = 1$. Recursion formulas were employed to derive the moments and central moments of the operators under consideration. These moments were subsequently utilized to determine the convergence rate of proposed operators in the sense of the usual modulus of continuity and Peetre's K-functional. Furthermore, the degree of approximation is also established for the function of bounded variation. We also illustrate via figures and tables that the proposed modification provides a better approximation for preserving the test function e_3.

Chapter 4 is divided into two sections. Pólya distribution-based generalization of λ-Bernstein operators is dealt with in the first section. For $f \in C[0,1]$, $\lambda \in [-1,1]$, $\mu = \mu(n) \to 0$ as $n \to \infty$, the generalization of λ-Bernstein operators [50] based on Pólya distribution is presented in the following manner:

$$\mathcal{P}_n^{\langle\lambda,\mu\rangle}(f;x) = \sum_{k=0}^{n} \hat{p}_{n,k}^{\langle\lambda,\mu\rangle}(x) f\left(\frac{k}{n}\right),$$

where $\hat{p}_{n,k}^{\langle\lambda,\mu\rangle}(x), k = 0, 1, ..., n$ are defined as:

$$\begin{cases} \hat{p}_{n,0}^{\langle\lambda,\mu\rangle}(x) = p_{n,0}^{\langle\mu\rangle}(x) - \frac{\lambda}{n+1} p_{n+1,1}^{\langle\mu\rangle}(x), \\ \hat{p}_{n,k}^{\langle\lambda,\mu\rangle}(x) = p_{n,k}^{\langle\mu\rangle}(x) + \lambda\left(\frac{n-2k+1}{n^2-1} p_{n+1,k}^{\langle\mu\rangle}(x) - \frac{n-2k-1}{n^2-1} p_{n+1,k+1}^{\langle\mu\rangle}(x)\right), 1 \leqslant k \leqslant n-1, \\ \hat{p}_{n,n}^{\langle\lambda,\mu\rangle}(x) = p_{n,n}^{\langle\mu\rangle}(x) - \frac{\lambda}{n+1} p_{n+1,n}^{\langle\mu\rangle}(x). \end{cases}$$

and

$$p_{n,k}^{\langle\mu\rangle}(x) = \begin{pmatrix} n \\ k \end{pmatrix} \frac{\prod_{i=0}^{k-1}(x + i\mu) \prod_{i=0}^{n-k-1}(1 - x + i\mu)}{\prod_{i=0}^{n-1}(1 + i\mu)}.$$

We establish some basic results that are relevant for establishing key theorems. We present a theorem and graphical illustrations in support of the proposed operator's interpolation behaviour. In order to verify the convergence of the proposed operators, we provide theoretical results alongside Mathematica-generated graphs.

The second section is concerned with the generalization of Bernstein operators with shifted knots. Shifted knots have the benefit of allowing approximation on interval $(0,1)$ as well as its subinterval. For $x \in \left(\frac{a_2}{n+b_2}, \frac{n+a_2}{n+b_2}\right)$, and $a_p,\ b_p,\ p = 1,2$ are positive real numbers with the condition $0 \leqslant a_2 \leqslant a_1 \leqslant b_1 \leqslant b_2$, we introduce the following generalization of Bernstein operators:

$$\mathbb{O}_n^{\langle a_i,b_i\rangle}(f;x) = \sum_{k=0}^{n} o_n^{\langle a_i,b_i\rangle}(x) f\left(\frac{k+a_1}{n+b_1}\right),$$

where

$$o_n^{\langle a_i,b_i\rangle}(x) = (n+b_2)\left(\frac{n+b_2}{n}\right)^{n-1}\left(\frac{k+a_2}{n+b_2} - x\right)^2\left(x - \frac{a_2}{n+b_2}\right)^{k-1}\left(\frac{n+a_2}{n+b_2} - x\right)^{n-k-1}.$$

We derive some theorems to establish the convergence of our newly constructed operators. To demonstrate asymptotic behaviour, we present Voronovskaja, and Grüss Voronovskaja type theorems. Finally, the convergence is verified using an absolute error table and graphical representations.

Chapter 5 is based on the bivariate generalization of operators involving a class of orthogonal polynomials called Apostol-Genocchi polynomials. Consider the set $C(I)$, which represents the class of real-valued continuous functions defined on the interval $I = [0, \infty] \times [0, \infty]$. For $f \in C(I)$, the bivariate generalization is defined as:

$$G_{n_1,n_2}^{\alpha,\beta}(f; x_1, x_2) = e^{-(n_1x_1+n_2x_2)}\left(\frac{1+e\beta}{2}\right)^{2\alpha}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{g_{k_1}^{\alpha}(n_1x_1;\beta)}{k_1!}\frac{g_{k_2}^{\alpha}(n_2x_2;\beta)}{k_2!}f\left(\frac{k_1}{n_1}, \frac{k_2}{n_2}\right),$$

The expression $g_k^{\alpha}(x;\beta)$ represents the generalized Apostol-Genocchi polynomials of order α. We obtain the rate of convergence in terms of partial and total modulus of continuity and order of approximation by means of a Lipschitz-type function and Peetre's K-functional. In addition, we propose a generalization known as "generalized boolean sum (GBS)" for these bivariate operators in order to determine the order of approximation for Bögel continuous functions. We utilize Mathematica software to present a few graphical illustrations that effectively showcase the rate of convergence for the bivariate operators. It gets known through those graphs that for some particular functions the bivariate operators exhibits superior convergence when $\alpha < \beta$. Based on our analysis and comparison of the error of approximation between the bivariate operators and the associated GBS operators, it can be concluded that the GBS operators exhibit a faster convergence towards the function.

The concluding chapter, **Chapter 6**, offers a summary of the thesis followed by the author's thoughts on the future direction of the research.
We now move on to our first chapter, which explores some important exponential operators based on recent studies by Ismail and May.

Chapter 2

On exponential operators due to Ismail and May

This chapter is dedicated to some exponential operators, a concept first introduced by May and later explored thoroughly in collaboration with Ismail. The operators are commonly known as Ismail-May operators among active researchers. The first section of this chapter is devoted to approximation properties of exponential operators linked to the polynomial $x(1+x)^2$. Further, we define its Bézier variant and estimate the rate of convergence for functions with derivatives of bounded variation, a direct approximation theorem using Ditizan-Totik modulus of smoothness, and a Voronovoskaya type result. Moreover, a two-variable generalization of the proposed operators and their approximation properties are also investigated. The second section is focused on another exponential operators associated with the polynomials $2x^{3/2}$. These operators are modified in a way that preserves the exponential function e^{-x} and provides improved approximations compared to the original operators for certain family of functions. We gave a result and supporting graphs to prove the goodness of modified operators.

2.1 General family of exponential operators

2.1.1 Introduction

In the year 1976, May [129] introduced the concept of positive exponential operators L_λ on $C(-\infty,\infty)$ (space of continuous functions defined on the entire real line) into C^∞

(space of infinitely differentiable functions) as:

$$L_\lambda(f;x) = \int_{-\infty}^{\infty} W(\lambda,t,s)f(s)ds,$$

where $W(\lambda,t,s) \geq 0$ is a kernel of distribution and satisfy the following conditions:

1. $L_\lambda(1;x) = \int_{-\infty}^{\infty} W(\lambda,t,s)ds = 1$ normalisation condition.

2. $\frac{\partial}{\partial t}W(\lambda,t,s) = \frac{\lambda}{p(t)}W(\lambda,t,s)(s-t)$, where $p(t)$ is analytic and positive for $t \in (-\infty,\infty)$.

The combination of the partial differential equation and the normalization condition establishes, at most, a single kernel $W(\lambda,t,u)$ for an exponential operator associated with a given polynomial $p(t)$. Furthermore, the normalization condition yields

$$\exp\left(\lambda \int_{c}^{g(x)} \frac{\theta d\theta}{p(\theta)}\right) = \int_{-\infty}^{\infty} C(\lambda,s)\exp(\lambda s x)\,ds, \quad x \in \text{Range of } q(t), \tag{2.1}$$

where $q(t) = \int_{c}^{t} \frac{dv}{p(v)}$ and $g(q(t)) = q(g(t))$.

Ismail and May [102] demonstrated that for a linear or quadratic $p(t)$, there exist several well-known operators, including Bernstein, Szász, Baskakov, Gauss-Weierstrass, Post-Widder etc., which satisfy above conditions and can therefore be classified as exponential operators. For example, when $p(t) = t$ and $c = 1$, the corresponding approximation operators are Szász operators:

$$S_\lambda(f;t) = e^{-\lambda t}\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{k!}f\left(\frac{k}{\lambda}\right), \quad t \in (0,\infty).$$

For $p(t) = 1$ and $c = 0$, equation (2.1) become the Gauss-Weierstrass operators:

$$W_\lambda(f;t) = \sqrt{\frac{\lambda}{2\pi}}\int_{-\infty}^{\infty}\exp\left\{\frac{-\lambda(s-t)^2}{2}\right\}f(s)ds, \quad t \in (-\infty,\infty).$$

Similarly, for a quadratic $p(t) = t(1-t)$ with $t \in (0,1)$ and $c = \frac{1}{2}$, equation (2.1) transforms into the very known Bernstein operators:

$$B_\lambda(f;t) = \sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix} t^k(1-t)^{n-k}f\left(\frac{k}{n}\right), \quad n = 1,2,.., \quad t \in [0,1].$$

Ismail and May [102] also constructed some new approximation operators for cubic polynomials by determining a unique generalized function $C(\lambda,s)$ for which corresponding kernel is given by:

$$W(\lambda,t,s) = \exp\left\{\lambda\int_{c}^{t}\frac{s-\theta}{p(\theta)}d\theta\right\}C(\lambda,s). \tag{2.2}$$

For $p(t) = t(1+t)^2$, $c = 1$ and considering the following identity (see [142])

$$e^{\lambda x} = \sum_{k=0}^{\infty} \frac{\lambda(\lambda+k)^{k-1}}{k!}(xe^{-x})^k, \tag{2.3}$$

we get

$$C(\lambda, s) = 2^{-\lambda s} e^{-\frac{\lambda(1+s)}{2}} \sum_{k=0}^{\infty} \frac{\lambda(\lambda+k)^{k-1}}{k!} \delta(k - \lambda s).$$

For this value of $C(\lambda, s)$, the corresponding exponential operators are

$$\widetilde{R_\lambda}(f;t) = e^{-\frac{\lambda t}{(1+t)}} \sum_{k=0}^{\infty} \frac{\lambda(\lambda+k)^{k-1}}{k!} \left(\frac{t}{1+t}\right)^k e^{-\frac{\lambda t}{(1+t)}} f\left(\frac{k}{\lambda}\right).$$

With change of variables $x = \frac{t}{1+t}$, Ismail and May [102] defined a positive linear operators for a continuous function $f \in [0,1]$, as follows:

$$R_\lambda(f;x) = e^{-\lambda x} \sum_{k=0}^{\infty} \frac{\lambda(\lambda+k)^{k-1}}{k!}(xe^{-x})^k f\left(\frac{k}{\lambda+k}\right). \tag{2.4}$$

The Kantorovich form of the operator (2.4) and its bivariate are introduced in [134], along with significant approximation results.

2.1.2 Basic Results

In this section, we discuss some auxiliary results that are essential to prove our main results for the operators (2.4).

Lemma 2.1.1 *For $e_i(s) = s^i$, $i = 0,1,2,3,4$ and $\lambda > 0$, we have*

$$\begin{aligned}
R_\lambda(e_0;x) &= 1;\\
R_\lambda(e_1;x) &= \frac{\lambda}{\lambda+1}x;\\
R_\lambda(e_2;x) &= \frac{\lambda^2}{(\lambda+1)(\lambda+2)}x^2 + \frac{\lambda}{(\lambda+1)^2}x;\\
R_\lambda(e_3;x) &= \frac{\lambda^3}{(\lambda+1)(\lambda+2)(\lambda+3)}x^3 + \frac{\lambda^2(3\lambda+4)}{(\lambda+1)^2(\lambda+2)^2}x^2 + \frac{\lambda}{(\lambda+1)^3}x;\\
R_\lambda(e_4;x) &= \frac{\lambda^4}{(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)}x^4 + \frac{2\lambda^3(3\lambda^2+11\lambda+9)}{(\lambda+1)^2(\lambda+2)^2(\lambda+3)^2}x^3\\
&\quad + \frac{\lambda^2(7\lambda^2+18\lambda+12)}{(\lambda+1)^3(\lambda+2)^3}x^2 + \frac{\lambda}{(\lambda+1)^4}x.
\end{aligned}$$

Lemma 2.1.2 *If $\mu_{\lambda,m} = R_\lambda((s-x)^m; x)$ denote the central moments of the operators (2.4), then for $m = 1, 2, 4$, we have*

$$\begin{aligned}
\mu_{\lambda,1}(x) &= -\frac{x}{(\lambda+1)};\\
\mu_{\lambda,2}(x) &= -\frac{(\lambda-2)}{(\lambda+1)(\lambda+2)}x^2 + \frac{\lambda}{(\lambda+2)}x;\\
\mu_{\lambda,4}(x) &= \frac{(3\lambda^2-46\lambda+24)x^4}{(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)} + \frac{\lambda(-6\lambda^3+36\lambda^2+216\lambda+216)x^3}{(\lambda+1)^2(\lambda+2)^2(\lambda+3)^2}\\
&\quad + \frac{\lambda(3\lambda^3-6\lambda^2-36\lambda-32)x^2}{(\lambda+1)^3(\lambda+2)^3} + \frac{\lambda x}{(\lambda+1)^4}.
\end{aligned}$$

Remark 2.1.3 *For sufficiently large λ, we have*

$$\lim_{\lambda\to\infty} \lambda\mu_{\lambda,1}(x) = -x;$$

$$\lim_{\lambda\to\infty} \lambda\mu_{\lambda,2}(x) = x(1-x);$$

$$\lim_{\lambda\to\infty} \lambda^2\mu_{\lambda,4}(x) = 3x^2(x-1)^2.$$

Lemma 2.1.4 *Let f be a continuous function in $[0,1]$, then we have*

$$|R_\lambda(f;x)| \leq \|f\|.$$

2.1.3 Main Results

Theorem 2.1.5 *Suppose f be a continuous function defined on the interval $[0,1]$, then $R_\lambda(f;x) \to f(x)$ uniformly in $[0,1]$.*

Proof: *From Lemma 2.1.1, $R_\lambda(e_0;x) = 1$, $R_\lambda(e_1;x) \to x$ and $R_\lambda(e_2;x) \to x^2$ as $\lambda \to \infty$. Then by Bohman-Korovkin theorem, $R_\lambda(f;x) \to f(x)$ uniformly in $[0,1]$.*

Now we estimate the rate of convergence of operators (2.4), with the help of first and second order modulus of continuity (see also [151]).

Theorem 2.1.6 *For $f \in C[0,1]$, $\lambda > 0$, we have*

$$|R_\lambda(f;x) - f(x)| \leq C\omega_2\left(f; \frac{\sqrt{\mu_{\lambda,2}(x)+\psi(x)}}{2}\right) + \omega(f;\psi^2(x)),$$

where $\psi(x) = \sqrt{\frac{x}{\lambda+1}}$ and $\mu_{\lambda,2}(x)$ is stated in Lemma 2.1.2.

Proof: We consider auxiliary operators

$$\widetilde{R_\lambda}(f;x) = R_\lambda(f;x) - f\left(\frac{\lambda x}{\lambda+1}\right) + f(x). \tag{2.5}$$

Suppose $g \in C^2[0,1]$, by Taylor's expansion, we have

$$g(s) = g(x) + g'(x)(s-x) + \int_x^s (s-u)g''(u)du.$$

From (2.5), $\widetilde{R_\lambda}(e_0; x) = 1$, $\widetilde{R_\lambda}(e_1; x) = x$ and $\widetilde{R_\lambda}((s-x); x) = 0$, we have

$$\begin{aligned}\left|\widetilde{R_\lambda}(g(s)-g(x);x)\right| &\leq \left|\int_x^{\frac{\lambda x}{\lambda+1}} \left(\frac{\lambda x}{\lambda+1} - u\right) g''(u)du\right| + \left|R_\lambda\left(\int_x^s (s-u)g''(u)du; x\right)\right| \\ &\leq \|g''\| \left(\int_x^{\frac{\lambda x}{\lambda+1}} \left|\frac{\lambda x}{\lambda+1} - u\right| du + R_\lambda\left(\int_x^s |(s-u)|du; x\right)\right) \\ &\leq \left\|g''\right\| (\mu_{\lambda,2}(x) + \psi(x)). \end{aligned} \tag{2.6}$$

From (2.5) and using Lemma 2.1.4, we get

$$\widetilde{R_\lambda}(f; x) \leq 3\|f\|. \tag{2.7}$$

Using equations (2.5) -(2.7), we have

$$\begin{aligned}|R_\lambda(f;x) - f(x)| \leq& \left|\widetilde{R_\lambda}(f(s)-g(s);x) - (f-g)(x)\right| + \left|\widetilde{R_\lambda}(g(s)-g(x);x)\right| + \left|f\left(\frac{\lambda x}{\lambda+1}\right) - f(x)\right| \\ \leq& 4\left[\|f-g\| + \frac{(\mu_{\lambda,2}(x)+\psi(x))}{4}\left\|g''\right\|\right] + \omega(f;\psi^2(x)). \end{aligned} \tag{2.8}$$

Taking infimum on the right hand side of (2.8), we have

$$|R_\lambda(f;x) - f(x)| \leq 4K_2\left(f; \frac{\mu_{\lambda,2}(x)+\psi(x)}{4}\right) + \omega(f;\psi^2(x)).$$

By using relation (1.2), we get the desired result .

In our next theorem, we discuss the rate of convergence for the operators R_λ using the functions of Lipschitz class as defined in subsection 1.1.7.

Theorem 2.1.7 *Let $f \in Lip_M(\beta)$, $x \in [0,1]$ and $\lambda > 0$, we have*

$$|R_\lambda(f;x) - f(x)| \leq M(\mu_{\lambda,2}(x))^{\frac{\beta}{2}}.$$

Proof: *For the positive linear operators R_λ and $f \in Lip_M(\beta)$, we have*

$$\begin{aligned}|R_\lambda(f;x) - f(x)| \leq& \left|\widetilde{R_\lambda}(|f(s)-f(x)|;x)\right| \\ =& e^{-\lambda x}\sum_{k=0}^{\infty}\frac{\lambda(\lambda+k)^{k-1}}{k!}(xe^{-x})^k\left|f\left(\frac{k}{\lambda+k}\right) - f(x)\right| \\ \leq& Me^{-\lambda x}\sum_{k=0}^{\infty}\frac{\lambda(\lambda+k)^{k-1}}{k!}(xe^{-x})^k\left|\frac{k}{\lambda+k} - x\right|^{\beta}. \end{aligned}$$

Using Hölder's inequality,

$$|R_\lambda(f;x) - f(x)| \leq M[\mu_{\lambda,2}(x)]^{\frac{\beta}{2}}.$$

Hence the proof.

Theorem 2.1.8 *If the function $f(x)$ is bounded on $[0,1]$, $x \in (0,1)$ and for which $f'(x)$, $f''(x)$ exist then, we have*

$$\lim_{\lambda\to\infty} \lambda\left[R_\lambda(f;x) - f(x)\right] = -xf'(x) + \frac{x(1-x)}{2} f''(x).$$

Proof: *Suppose $x \in [0,1]$ be a fixed point, by Taylor's formula, we can say*

$$f(s) = f(x) + (s-x)f'(x) + \frac{(s-x)^2}{2} f''(x) + r(s,x)(s-x)^2, \tag{2.9}$$

$r(s,x) \in C[0,1]$ be the Peano form of the remainder. By using L'hospital rule we can easily say that $r(s,x)$ converges to 0 when s approaches to x.
In (2.9), applying $R_\lambda(.,x)$, we obtain

$$\begin{aligned}\lim_{\lambda\to\infty} \lambda\left[R_\lambda(f;x) - f(x)\right] =& f'(x)\lim_{\lambda\to\infty} \lambda R_\lambda((s-x);x) + \frac{f''(x)}{2}\lim_{\lambda\to\infty} \lambda R_\lambda\left((s-x)^2;x\right)\\ &+ \lim_{\lambda\to\infty} \lambda R_\lambda\left(r(s,x)(s-x)^2;x\right).\end{aligned} \tag{2.10}$$

Applying Cauchy-Schwarz inequality in the last term of (2.10), we get

$$\lambda R_\lambda\left((s-x)^2;x\right) \leq \sqrt{R_\lambda(r^2(s,x);x)}.\sqrt{\lambda^2 R_\lambda((s-x)^4;x)}.$$

Since $r^2(x,x) = 0$ and from Remark 2.1.3, we obtain

$$\lim_{\lambda\to\infty} \lambda R_\lambda\left(r(s,x)(s-x)^2;x\right) = 0. \tag{2.11}$$

From (2.10), (2.11) and using Remark 2.1.3, we get the required result.

From [74] the unified Ditzian-Totik modulus of smoothness is given as follows:

$$\omega_{\phi^\tau}(f;t) = \underset{0<h\leq t}{Sup}\left\{\left|f\left(x + \frac{h\phi^\tau(x)}{2}\right) - f\left(x - \frac{h\phi^\tau(x)}{2}\right)\right|, x + \frac{h\phi^\tau(x)}{2} \in [0,1]\right\}.$$

Further, the appropriate K-functional is defined by

$$K_{\phi^\tau}(f;t) = \inf_{g\in W_{\phi^\tau[0,1]}} \{\|f-g\| + t\|\phi^\tau g'\|\} \qquad (t>0),$$

where $0 \leq \tau \leq 1$, $W_{\phi^\tau}[0,1] = \{g : g \in AC_{loc}[0,1], \|\phi^\tau g'\| < \infty\}, g \in AC_{loc}[0,1]$ denotes the class of all locally absolutely continuous function and $\|.\|$ is the sup norm on $C[0,1]$. It is well known [74] that there exists a constant $C > 0$ such that

$$C^{-1}\omega_{\phi^\tau}(f;t) \leq K_{\phi^\tau}(f;t) \leq C\omega_{\phi^\tau}(f;t). \tag{2.12}$$

Theorem 2.1.9 *Let $f \in C[0,1]$. Then for $\phi(x) = \sqrt{x(1-x)}$ and for every $x \in (0,1)$, we have*

$$|R_\lambda(g;x) - g(x)| \le C\omega_{\phi^\tau}\left(f; \frac{\phi^{1-\tau}(x)}{\sqrt{\lambda}}\right),$$

where C is a constant.

Proof: Since $g \in W_{\phi^\tau}$, we obtain

$$g(s) = g(x) + \int_x^s g'(u)du,$$

Therefore, we can write

$$|R_\lambda(g;x) - g(x)| \le R_\lambda\left(\left|\int_x^s g'(u)du\right|; x\right). \tag{2.13}$$

Applying Hölder's inequality, we get

$$\left|\int_x^s g'(u)du\right| \le \|\phi^\tau g'\| \left|\int_x^s \frac{du}{\phi^\tau}\right| \le \|\phi^\tau g'\| \, |s-x|^{1-\tau} \left|\int_x^s \frac{du}{\phi(u)}\right|^\tau \tag{2.14}$$

and

$$\begin{aligned}
\left|\int_x^s \frac{du}{\phi(u)}\right| &= \left|\int_x^s \frac{du}{\sqrt{u(1-u)}}\right| \\
&\le \left|\int_x^s \left(\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{1-u}}\right) du\right| \\
&\le 2[\left|\sqrt{s} - \sqrt{x}\right| + \left|\sqrt{1-s} - \sqrt{1-x}\right|] \\
&= 2\,|s-x| \left[\frac{1}{\sqrt{s} + \sqrt{x}} + \frac{1}{\sqrt{1-s} + \sqrt{1-x}}\right] \\
&\le 2\,|s-x| \left[\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x}}\right] \\
&\le \frac{2\sqrt{2}\,|s-x|}{\phi(x)}.
\end{aligned}$$

From (2.14), we have

$$\int_x^s g'(u)du \le \|\phi^\tau g'\| \, |s-x|^{1-\tau} \left(\frac{2\sqrt{2}\,|s-x|}{\phi(x)}\right)^\tau = \frac{\|\phi^\tau g'\| \, |s-x| \, 2^\tau 2^{\frac{\tau}{2}}}{\phi^\tau(x)}.$$

Applying Cauchy-Schwarz inequality,

$$\begin{aligned}|R_\lambda(g;x)-g(x)| &\leq \frac{2^\tau 2^{\frac{\tau}{2}}\,\|\phi^\tau g'\|\,R_\lambda(|s-x|\,;x)}{\phi^\tau(x)}\\ &\leq \frac{2^{\frac{3\tau}{2}}\,\|\phi^\tau g'\|\,\sqrt{R_\lambda\left((s-x)^2;x\right)}}{\phi^\tau(x)}\\ &\leq \frac{2^{\frac{3\tau}{2}}\,\|\phi^\tau g'\|}{\phi^\tau(x)}\frac{C_1\phi(x)}{\sqrt{\lambda}}=\frac{C_1 2^{\frac{3\tau}{2}}\,\|\phi^\tau g'\|\,\phi^{1-\tau}(x)}{\sqrt{\lambda}}.\end{aligned}$$

Therefore

$$\begin{aligned}|R_\lambda(f;x)-f(x)| &\leq |R_\lambda((f-g);x)|+|R_\lambda(g;x)-g(x)|+|g(x)-f(x)|\\ &\leq 2\,\|f-g\|+|R_\lambda(g;x)-g(x)|\\ &\leq 2\,\|f-g\|+\frac{C_1 2^{\frac{3\tau}{2}}\,\|\phi^\tau g'\|\,\phi^{1-\tau}(x)}{\sqrt{\lambda}}\\ &\leq C\left\{\|f-g\|+\frac{\phi^{1-\tau}(x)}{\sqrt{\lambda}}\,\|\phi^\tau g'\|\right\}\leq CK_{\phi^\tau}\left(f,\frac{\phi^{1-\tau}(x)}{\sqrt{\lambda}}\right),\end{aligned}$$

where $C=max\left\{2,C_1 2^{\frac{3\tau}{2}}\right\}$.
Using relation (2.12), we get required result.

Example 2.1.10 *For $\lambda=10,20,100$ the rate of convergence of the operators R_λ to the function $f(x)=9x^2-6x+6/5$ is illustrated in Fig 2.1. Further, in Table 2.1, we have estimated the absolute error $E_\lambda=|R_\lambda(f;x)-f(x)|$ for different values of λ and given the corresponding graph for error depicting the convergence in Fig 2.2 . It can be clearly seen from Fig 2.1, Fig 2.2 and from the Table 2.1 that for larger values of λ the proposed operators (2.4) converges to $f(x)$.*

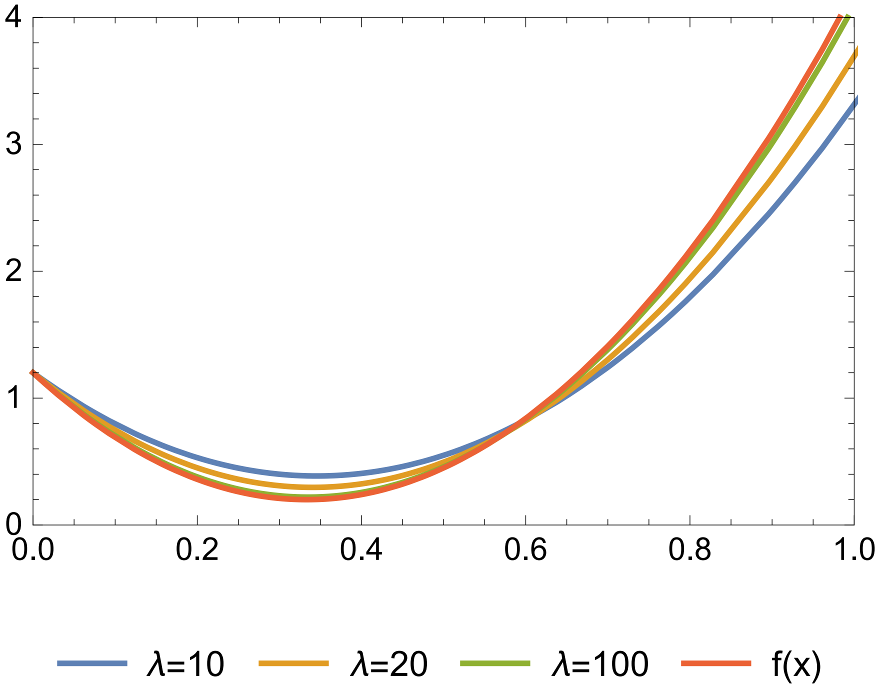


Figure 2.1: The convergence of operators R_λ to the function $f(x) = 9x^2 - 6x + 6/5$ for $\lambda = 10, 20, 100$.

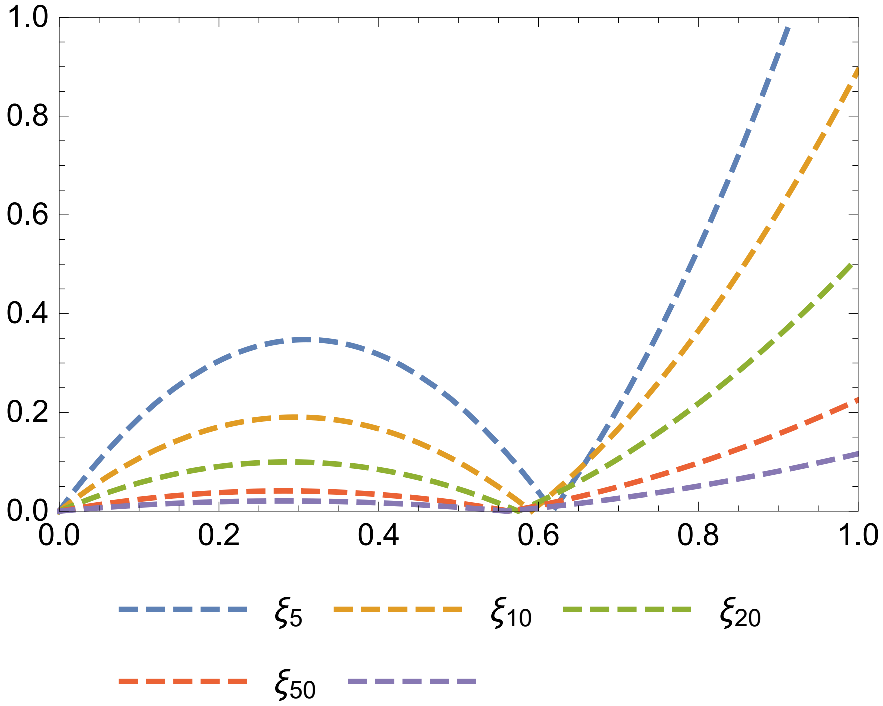


Figure 2.2: Graphical representation of absolute error of operators R_λ to the function $f(x) = 9x^2 - 6x + 6/5$ for $\lambda = 5, 10, 20, 50, 100$.

Table 2.1: Error of approximation process for $f(x) = 9x^2 - 6x + 6/5$.

x	E_5	E_{10}	E_{20}	E_{50}	E_{100}
0.08	0.156686	0.089180	0.047780	0.019951	0.010122
0.16	0.266743	0.150426	0.080101	0.033300	0.016867
0.24	0.330171	0.183749	0.096962	0.040046	0.020235
0.32	0.346971	0.189144	0.098363	0.040189	0.020226
0.40	0.317143	0.166612	0.084304	0.033729	0.016834
0.48	0.240686	0.116152	0.054786	0.020667	0.010077
0.56	0.117600	0.037765	0.009808	0.001002	0.000064
0.64	0.052114	0.068549	0.050630	0.025266	0.013581
0.72	0.268457	0.202790	0.126528	0.058137	0.030475
0.80	0.531429	0.364959	0.217885	0.097610	0.050747
0.88	0.841029	0.555055	0.324702	0.143686	0.074395
0.96	1.197260	0.773078	0.446979	0.196364	0.101421

2.1.4 Bézier Variant of Ismail-May Operators

Zeng and Piriou [174] in the year 1997, constructed the Bernstein-Bézier type operators and studied its rate of convergence for bounded variation functions. Gupta et.al. [97] proposed the Bézier variant of the Szász-Kantorovich operators and investigated a convergence theorem for locally bounded functions subsuming the approximation of functions of bounded variation as a special case.

Motivated by the above stated work, this section presents the formal definition of the Bézier variant of the operators (2.4) in the following manner:

$$R_\lambda^\alpha(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{\lambda+k}\right)\vartheta_{\lambda,k}^\alpha(x), \quad \alpha \geq 1, \;\; x \in [0,1], \tag{2.15}$$

where $\vartheta_{\lambda,k}^\alpha(x) = (P_{\lambda,k}(x))^\alpha - (P_{\lambda,k+1}(x))^\alpha$, and $P_{\lambda,k}(x) = \sum\limits_{j=k}^{\infty} r_{\lambda,j}(x)$ for $k = 0, 1, 2, ..n$ are the Bézier basis functions and

$$r_{\lambda,k}(x) = e^{-\lambda x}\frac{\lambda(\lambda+k)^{k-1}}{k!}(xe^{-x})^k, \quad x \in [0,1].$$

Lemma 2.1.11 *Let $f \in C[0,1]$. Then Bézier variant of the operators derived from the Ismail-May operators exhibit the following property:*

(i) $\left\|R_\lambda^\alpha\right\| \leq \|f\|$;

(ii) $R_\lambda^\alpha(f;x) \leq \alpha R_\lambda(f;x)$.

Proof:

(i) $R_\lambda^\alpha(e_0;x) = \sum_{k=0}^{\infty} 1.\vartheta_{\lambda,k}^\alpha(x) = 1,$

it follows that

$$\left|R_\lambda^\alpha(f;x)\right| \leq \|f\| \sum_{k=0}^{\infty} \vartheta_{\lambda,k}^\alpha(x) = \|f\|.$$

(ii) Using the inequality $|a^\alpha - b^\alpha| \leq \alpha\,|a-b|$ where $0 \leq a, b \leq 1$, and $\alpha \geq 1$, we get

$$\begin{aligned} 0 &\leq (P_{\lambda,k}(x))^\alpha - (P_{\lambda,k+1}(x))^\alpha \\ &\leq \alpha\,(P_{\lambda,k}(x) - P_{\lambda,k+1}(x)) \\ &= \alpha r_{\lambda,k}(x). \end{aligned}$$

Hence in view of definition of R_λ^α and the positivity of f, we get the result.

2.1.5 Global Approximation Theorem

Now, we present a global approximation theorem for the operators (2.15) using the first order Ditzian Totik modulus of smoothness defined in subsection 1.1.4.

Theorem 2.1.12 *Let $f \in [0,1]$ and $\phi(x) = \sqrt{x(1-x)}$. For every $x \in [0,1)$ and sufficiently large λ, we have*

$$\left|R_\lambda^\alpha(f;x) - f(x)\right| \leq C\omega_\phi\left(f;\frac{1}{\sqrt{\lambda}}\right).$$

Proof: By definition of $K_\phi(f,t)$, for fixed λ, x we can choose $g = g_{\lambda,x} \in W_\phi[0,1)$ such that

$$\|f-g\| + \frac{1}{\sqrt{\lambda}}\|\phi g'\| + \frac{1}{\lambda}\|g'\| \leq \omega_\phi\left(f;\frac{1}{\sqrt{\lambda}}\right). \tag{2.16}$$

Then

$$\begin{aligned} \left|R_\lambda^\alpha(f;x) - f(x)\right| &\leq \left|R_\lambda^\alpha(f-g;x)\right| + |f-g| + \left|R_\lambda^\alpha(g;x) - g(x)\right| \\ &\leq C\,\|f-g\| + \left|R_\lambda^\alpha(g;x) - g(x)\right|. \end{aligned}$$

In order to calculate the relation mentioned above, the domain is divided into two subintervals, $x \in I_\lambda = [0,\frac{1}{\lambda}]$ and $x \in I_\lambda^c = (\frac{1}{\lambda},1)$.

Using the representation

$$g(s) = g(x) + \int_x^s g'(z)dz,$$

we can write

$$\left|R_\lambda^\alpha(g;x) - g(x)\right| \leq \left|R_\lambda^\alpha\left(\int_x^s g'(z)dz;x\right)\right|. \tag{2.17}$$

Let $x \in I_\lambda^c = (\frac{1}{\lambda}, 1)$, we have

$$\begin{aligned}\left|\int_x^s g'(z)dz\right| &\leq \|\phi g'\| \left|\int_x^s \frac{1}{\phi(z)}dz\right| \\ &\leq \|\phi g'\| \left|\int_x^s \frac{1}{\sqrt{z(1-z)}}dz\right| \\ &\leq \|\phi g'\| \frac{2\sqrt{2}\,|s-x|}{\phi(x)}. \qquad (2.18)\end{aligned}$$

By combining (2.17) and (2.18), we have

$$\begin{aligned}\left|R_\lambda^\alpha(g;x) - g(x)\right| &\leq \frac{2\sqrt{2}\,\|\phi g'\|}{\phi(x)} \left|R_\lambda^\alpha(|s-x|;x)\right| \\ &\leq \frac{2\sqrt{2}\,\|\phi g'\|}{\phi(x)} \left|\left(R_\lambda^\alpha\left((s-x)^2;x\right)\right)^{1/2}\right| \\ &\leq \frac{2\sqrt{2\alpha}\,\|\phi g'\|}{\phi(x)} \sqrt{\frac{cx(1-x)}{\lambda}} \\ &\leq C\frac{\|\phi g'\|}{\sqrt{\lambda}}.\end{aligned}$$

Again for $x \in I_\lambda = [0, \frac{1}{\lambda}]$, using Lemma 2.1.11 and Remark 2.1.3

$$\begin{aligned}\left|R_\lambda^\alpha(g;x) - g(x)\right| &\leq \|g''\| \left|R_\lambda^\alpha(|s-x|;x)\right| \\ &\leq \sqrt{\alpha}\,\|g''\| \left(R_\lambda^\alpha\left((s-x)^2;x\right)\right)^{\frac{1}{2}} \\ &\leq \sqrt{\alpha}\,\|g''\| \sqrt{\frac{cx(1-x)}{\lambda}} \\ &\leq \frac{C\,\|g''\|}{\lambda}.\end{aligned}$$

Therefore,

$$\left|R_\lambda^\alpha(g;x) - g(x)\right| \leq C\left(\frac{\|\phi g''\|}{\sqrt{\lambda}} + \frac{\|g'\|}{\lambda}\right). \qquad (2.19)$$

Collecting equations (2.16)-(2.19), we obtain the required result.

2.1.6 Voronovoskaya Theorem

In this section, we present a quantitative Voronovoskaya theorem for operators (2.15) involving Ditizan Totik modulus of smoothness (see also [154]).

Theorem 2.1.13 *Let $f \in C^2[0,1], x \in [0,1)$. For sufficiently large λ the following inequality hold*

$$\left|\lambda\left[R_\lambda^\alpha(f;x) - f(x) - \mu_{\lambda,\alpha}^{(1)}(x)f'(x) - \frac{1}{2}\mu_{\lambda,\alpha}^{(2)}(x)f''(x)\right]\right| \leq \begin{cases} C\omega_\phi(f'';\lambda^{-1/2}\phi(x)) \\ C\phi(x)\omega_\phi(f'';\lambda^{-1/2}), \end{cases}$$

where $\mu_{\lambda,\alpha}^{(n)}(x) = R_\lambda^\alpha((s-x)^n;x)$.

Proof: Let $f \in C^2[0,1]$ be given and $s, x \in [0,1]$. Using Taylor's expansion

$$f(s) - f(x) = (s-x)f'(x) + \int_x^s (s-u)f''(u)du,$$

we get

$$f(s) - f(x) - (s-x)f'(x) - \frac{1}{2}(s-x)^2 f''(x) = \int_x^s (s-u)f''(u)du - \int_x^s (s-u)f''(x)du$$
$$= \int_x^s (s-u)[f''(u) - f''(x)]du.$$

Applying R_λ^α to both sides of the above relation, we obtain

$$\left|R_\lambda^\alpha(f;x) - f(x) - \mu_{\lambda,\alpha}^{(1)}(x)f'(x) - \frac{1}{2}\mu_{\lambda,\alpha}^{(2)}(x)f''(x)\right| \leq R_\lambda^\alpha\left(\left|\int_x^s |s-u|\,|f''(u) - f''(x)|\,du\right|;x\right). \tag{2.20}$$

From [79], for $g \in W_\phi[0,1]$ the following estimates can be obtained,

$$\left|\int_x^s |s-u|\,|f''(u) - f''(x)|\,du\right| \leq 2\,\|f'' - g\|\,(s-x)^2 + 2\,\|\phi g'\|\,\phi^{-1}(x)|s-x|^3. \tag{2.21}$$

Using relations (2.20)-(2.21), Lemma 2.1.11, Remark 2.1.3 and Cauchy-Schwarz inequality, we have

$$\left|R_\lambda^\alpha(f;x) - f(x) - \mu_{\lambda,\alpha}^{(1)}(x)f'(x) - \frac{1}{2}\mu_{\lambda,\alpha}^{(2)}(x)f''(x)\right|$$
$$\leq 2\,\|f'' - g\|\,R_\lambda^\alpha\left((s-x)^2;x\right) + 2\,\|\phi g'\|\,\phi^{-1}(x)R_\lambda^\alpha\left(|s-x|^3;x\right)$$
$$\leq 2\,\|f'' - g\|\,\alpha R_\lambda\left((s-x)^2;x\right) + 2\alpha\,\|\phi g'\|\,\phi^{-1}(x)\left\{R_\lambda(s-x)^2;x\right\}^{1/2}\left\{R_\lambda(s-x)^4;x\right\}^{1/2}$$
$$\leq 2\,\|f'' - g\|\,\alpha R_\lambda\left((s-x)^2;x\right) + \frac{2\alpha C}{\lambda}\,\|\phi g'\|\left\{R_\lambda(s-x)^2;x\right\}^{1/2}$$
$$\leq 2\,\|f'' - g\|\,\alpha\frac{Cx(1-x)}{\lambda} + \frac{2\alpha C}{\lambda}\,\|\phi g'\|\,\sqrt{\frac{Cx(1-x)}{\lambda}}$$
$$\leq \frac{C}{\lambda}\left\{\phi^2(x)\,\|f'' - g\| + \lambda^{-1/2}\phi(x)\,\|\phi g'\|\right\},$$

where the constant $C > 0$ is not the same at each occurrence.
Since $\phi^2(x) \leq \phi(x) \leq 1$ and $x \in [0,1)$, we obtain

$$\left| R_\lambda^\alpha(f;x) - f(x) - \mu_{\lambda,\alpha}^{(1)}(x)f'(x) - \frac{1}{2}\mu_{\lambda,\alpha}^{(2)}(x)f''(x) \right| \\ \leq \frac{C}{\lambda}\left\{ \|f''-g\| + \lambda^{-1/2}\phi(x)\,\|\phi g'\| \right\}. \quad (2.22)$$

The above inequality can be rewritten as

$$\left| R_\lambda^\alpha(f;x) - f(x) - \mu_{\lambda,\alpha}^{(1)}(x)f'(x) - \frac{1}{2}\mu_{\lambda,\alpha}^{(2)}(x)f''(x) \right| \\ \leq \frac{C\phi(x)}{\lambda}\left\{ \|f''-g\| + \lambda^{-1/2}\,\|\phi g'\| \right\}. \quad (2.23)$$

Taking infimum on RHS of (2.22) and (2.23) and for $g \in W_\phi[0,1]$, we have

$$\left| \lambda \left[R_\lambda^\alpha(f;x) - f(x) - \mu_{\lambda,\alpha}^{(1)}(x)f'(x) - \frac{1}{2}\mu_{\lambda,\alpha}^{(2)}(x)f''(x) \right] \right| \leq \begin{cases} CK_\phi\left(f''; \lambda^{-1/2}\phi(x)\right) \\ C\phi(x)K_\phi\left(f''; \lambda^{-1/2}\right). \end{cases}$$

2.1.7 Functions with Derivative of Bounded Variation

Let $DBV[0,1]$ be the space of all absolutely continuous functions f defined on [0,1] and having a derivative f' equivalent with a function of bounded variation on $[0,1]$. For $f \in DBV[0,1]$ we may write,

$$f(x) = \int_0^x g(t)\,dt + f(0).$$

We can rewrite the operators given by (2.15) as

$$R_\lambda^\alpha(f;x) = \int_0^1 f(s)\frac{\partial}{\partial s}\left\{ M_\lambda^\alpha(x,s) \right\} dw,$$

where

$$M_\lambda^\alpha(x,s) = \left\{ \begin{array}{cc} \sum\limits_{\frac{k}{k+\lambda} \leq s} Q_{\lambda,k}^\alpha(x) & ,0 < s < 1 \\ 0 & ,s = 0. \end{array} \right\}.$$

The subsequent Lemma is referenced as a prerequisite for establishing the main theorem (see also [152]).

Lemma 2.1.14 *Let $x \in [0,1)$, then for sufficiently large n, we have*

(i) $\vartheta_\lambda^\alpha(x,s) \leq \frac{C\alpha x(1-x)}{\lambda(x-s)^2}$;

(ii) $1 - \vartheta_\lambda^\alpha(x,z) \leq \frac{C\alpha x(1-x)}{\lambda(x-z)^2}$.

Proof:

(i) $$\vartheta_\lambda^\alpha(x,s)=\int_0^s \frac{\partial}{\partial v}\{\mathcal{M}_\lambda^\alpha(x,v)\}dv \le \int_0^s \left(\frac{x-v}{x-s}\right)^2 \frac{\partial}{\partial v}\{\mathcal{M}_\lambda^\alpha(x,v)\}dv$$
$$=\frac{1}{(x-s)^2}(R_\lambda^\alpha(x-v)^2)(x,v) \le \frac{C\alpha x(1-x)}{\lambda(x-s)^2}.$$

(ii) The proof of this part is left to the readers which is similar to part (i).

In the theorem given below, we study the rate of convergence for functions with derivative of bounded variation (see [72]).

Theorem 2.1.15 *Let $f \in DBV[0,1]$. Then, we have*

$$\left|R_\lambda^\alpha(f;x)-f(x)\right| \le \left|\frac{(f'(x+)+\alpha f'(x-))}{\alpha+1}\right| \sqrt{\frac{C\alpha x(1-x)}{\lambda}} + |(f'(x+)+\alpha f'(x-))| \sqrt{\frac{C\alpha x(1-x)}{\lambda}}$$
$$+ \frac{C\alpha(1-x)}{\lambda}\sum_{k=1}^{[\sqrt{\lambda}]} \bigvee_{x-x/k}^{x} f_x{}' + \frac{x}{\sqrt{\lambda}} \bigvee_{x-x/\sqrt{\lambda}}^{x} f_x{}' + \frac{1-x}{\sqrt{\lambda}}\left(\bigvee_{x}^{x+\frac{1-x}{\sqrt{\lambda}}} f_x{}'\right) + \frac{C\alpha x}{\lambda}\sum_{k=1}^{[\sqrt{\lambda}]}\left(\bigvee_{x}^{x+\frac{1-x}{k}} f_x{}'\right),$$

where

$$f_x{}'(s)=\begin{cases} f'(s)-f'(x-), & 0\le s<x\\ 0 & s=x\\ f'(s)-f'(x+), & x<s<\infty. \end{cases} \tag{2.24}$$

Proof: For $f \in DBV[0,1]$, we may write

$$f'(s)=f_x{}'(s)+\frac{1}{\alpha+1}(f'(x+)+\alpha f'(x-))+\frac{1}{2}(f'(x+)-f'(x-))\left(\operatorname{sgn}(s-x)+\frac{\alpha-1}{\alpha+1}\right)$$
$$+\delta_x(s)\left[f'(s)-\frac{1}{2}(f'(x+)+f'(x-))\right], \tag{2.25}$$

where

$$\delta_x(s)=\begin{cases} 1 & s=x\\ 0, & s\neq x. \end{cases}$$

Again, we have

$$
\begin{aligned}
&R_\lambda^\alpha(f;x) - f(x)\\
&= \int_0^1 (f(s) - f(x))\frac{\partial}{\partial s}\{\mathcal{M}_\lambda^\alpha(x,s)\}\,ds\\
&= \int_0^x (f(s) - f(x))\frac{\partial}{\partial s}\{\mathcal{M}_\lambda^\alpha(x,s)\}\,ds + \int_x^1 (f(s) - f(x))\frac{\partial}{\partial s}\{\mathcal{M}_\lambda^\alpha(x,s)\}\,ds\\
&= -\int_0^x \left(\int_s^x f'(u)du\right)\frac{\partial}{\partial s}\{\mathcal{M}_\lambda^\alpha(x,s)\}\,ds + \int_x^1 \left(\int_s^x f'(u)du\right)\frac{\partial}{\partial s}\{\mathcal{M}_\lambda^\alpha(x,s)\}\,ds\\
&= -A_\lambda^\alpha(x) + B_\lambda^\alpha(x). \qquad (2.26)
\end{aligned}
$$

Now, from equation (2.25), we have

$$
\begin{aligned}
A_\lambda^\alpha(x) = {}& \frac{(f'(x+) + \alpha f'(x-))}{\alpha+1}\int_0^x (x-s)\,\frac{\partial}{\partial s}\{\mathcal{M}_\lambda^\alpha(x,s)\}\,ds + \int_0^x \left(\int_s^x f_x{}'(u)du\right)\frac{\partial}{\partial s}\{\mathcal{M}_\lambda^\alpha(x,s)\}\,ds\\
&- \frac{2}{\alpha+1}\frac{f'(x+) - f'(x-)}{2}\int_0^x (x-s)\,\frac{\partial}{\partial s}\{\mathcal{M}_\lambda^\alpha(x,s)\}\,ds. \qquad (2.27)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
B_\lambda^\alpha(x) = {}& \frac{(f'(x+) + \alpha f'(x-))}{\alpha+1}\int_x^1 (s-x)\,\frac{\partial}{\partial s}\{\mathcal{M}_\lambda^\alpha(x,s)\}\,ds + \int_x^1 \left(\int_x^s f_x{}'(u)du\right)\frac{\partial}{\partial s}\{\mathcal{M}_\lambda^\alpha(x,s)\}\,ds\\
&+ \frac{2\alpha}{\alpha+1}\frac{f'(x+) - f'(x-)}{2}\int_x^1 (s-x)\,\frac{\partial}{\partial s}\{\mathcal{M}_\lambda^\alpha(x,s)\}\,ds. \qquad (2.28)
\end{aligned}
$$

Using (2.27)-(2.28)and from (2.25), we get

$$
\begin{aligned}
&R_\lambda^\alpha(f;x) - f(x)\\
&= \frac{(f'(x+) + \alpha f'(x-))}{\alpha+1}\int_0^1 (s-x)\,\frac{\partial}{\partial s}\{\mathcal{M}_\lambda^\alpha(x,s)\}\,ds + \frac{2}{\alpha+1}\frac{f'(x+) - f'(x-)}{2}\\
&\times \int_0^x (x-s)\,\frac{\partial}{\partial s}\{\mathcal{M}_\lambda^\alpha(x,s)\}\,ds + \frac{2\alpha}{\alpha+1}\frac{f'(x+) - f'(x-)}{2}\int_x^1 (s-x)\,\frac{\partial}{\partial s}\{\mathcal{M}_\lambda^\alpha(x,s)\}\,ds\\
&- \int_0^x \left(\int_s^x f_x{}'(u)du\right)\frac{\partial}{\partial s}\{\mathcal{M}_\lambda^\alpha(x,s)\}\,ds + \int_x^1 \left(\int_x^s f_x{}'(u)du\right)\frac{\partial}{\partial s}\{\mathcal{M}_\lambda^\alpha(x,s)\}\,ds.
\end{aligned}
$$

Hence

$$\left|R_\lambda^\alpha(f;x)-f(x)\right| \le \left|\frac{(f'(x+)+\alpha f'(x-))}{\alpha+1}\right| (R_\lambda^\alpha|s-x|)(x) + |f'(x+)-f'(x-)|\left|R_\lambda^\alpha|s-x|)(x)\right|$$
$$+\left|\int_0^x\left(\int_s^x f_x'(u)du\right)\frac{\partial}{\partial s}\{\mathcal{M}_\lambda^\alpha(x,s)\}ds\right| + \left|\int_x^1\left(\int_x^s f_x'(u)du\right)\frac{\partial}{\partial s}\{\mathcal{M}_\lambda^\alpha(x,s)\}ds\right|.$$

Applying Cauchy-Schwarz inequality, we get

$$\left|R_\lambda^\alpha(f;x)-f(x)\right| \le \left|\frac{(f'(x+)+\alpha f'(x-))}{\alpha+1}\right| \sqrt{(R_{\lambda,\alpha}(s-x)^2)(x)} + |f'(x+)-f'(x-)|\sqrt{(R_{\lambda,\alpha}(s-x)^2)(x)}$$
$$+\left|\int_0^x\left(\int_s^x f_x'(u)du\right)\frac{\partial}{\partial s}\{\mathcal{M}_\lambda^\alpha(x,s)\}ds\right| + \left|\int_x^1\left(\int_x^s f_x'(u)du\right)\frac{\partial}{\partial s}\{\mathcal{M}_\lambda^\alpha(x,s)\}ds\right|. \tag{2.29}$$

Now, using Lemma 2.1.1 and integration by parts, we get

$$\int_0^x\left(\int_s^x f_x'(u)du\right)\frac{\partial}{\partial s}\{\mathcal{M}_\lambda^\alpha(x,s)\}ds = \int_0^x\left(\int_s^x f_x'(u)du\right)\frac{\partial}{\partial s}\{\vartheta_\lambda^\alpha(x,s)\}ds = -\int_0^x f_x'(s)\vartheta_\lambda^\alpha(x,s)ds.$$

Therefore

$$\left|\int_0^x\left(\int_s^x f_x'(u)du\right)\frac{\partial}{\partial s}\{\mathcal{M}_n^\alpha(x,s)\}ds\right| \le \int_0^x \left|f_x'(s)\right|\vartheta_n^\alpha(x,s)ds$$
$$\le \int_0^{x-\frac{x}{\sqrt{\lambda}}} \left|f_x'(s)\right|\vartheta_n^\alpha(x,s)ds + \int_{x-\frac{x}{\sqrt{\lambda}}}^x \left|f_x'(s)\right|\vartheta_n^\alpha(x,s)ds.$$

Using $f_x'(x)=0$ and $\vartheta_\lambda^\alpha(x,s)\le 1$, we get

$$\int_{x-\frac{x}{\sqrt{\lambda}}}^x \left|f_x'(s)\right|\vartheta_\lambda^\alpha(x,s)ds = \int_{x-\frac{x}{\sqrt{\lambda}}}^x \left|f_x'(s)-f_x'(x)\right|\vartheta_\lambda^\alpha(x,s)ds$$
$$\le \int_{x-\frac{x}{\sqrt{\lambda}}}^x \bigvee_s^x (f_x')ds \le \bigvee_{x-x/\sqrt{\lambda}}^x f_x' \int_{x-\frac{x}{\sqrt{\lambda}}}^x \bigvee_s^x f_x' ds = \frac{x}{\sqrt{\lambda}} \bigvee_{x-x/\sqrt{\lambda}}^x f_x'.$$

Again, using $\vartheta_\lambda^\alpha(x,s)\le \frac{C\alpha x(1-x)}{\lambda(x-s)^2}$ and putting $s = x-\frac{x}{u}$, we get

$$\int_0^{x-\frac{x}{\sqrt{\lambda}}} \left|f_x'(s)\right|\vartheta_\lambda^\alpha(x,s)ds \le \frac{C\alpha(1-x)}{\lambda}\int_1^{\sqrt{\lambda}} \bigvee_{x-x/\sqrt{\lambda}}^x f_x' du \le \frac{C\alpha(1-x)}{\lambda}\sum_{k=1}^{[\sqrt{\lambda}]}\bigvee_{x-x/k}^x f_x'.$$

Hence

$$\left|\int_0^x\left(\int_s^x f_x'(u)du\right)\frac{\partial}{\partial s}\{\mathcal{M}_\lambda^\alpha(x,s)\}ds\right| \le \frac{C\alpha(1-x)}{\lambda}\sum_{k=1}^{[\sqrt{\lambda}]}\bigvee_{x-x/k}^x f_x' + \frac{x}{\sqrt{\lambda}}\bigvee_{x-x/\sqrt{\lambda}}^x f_x'. \tag{2.30}$$

Now

$$\begin{aligned}
&\left|\int_x^1\left(\int_x^s f_x'(u)du\right)\frac{\partial}{\partial s}\{\mathcal{M}_\lambda^\alpha(x,s)\}\,ds\right|\\
&=\left|\int_x^z\left(\int_x^s f_x'(u)du\right)\frac{\partial}{\partial s}\{1-\vartheta_\lambda^\alpha(x,s)\}\,ds+\int_z^1\left(\int_x^s f_x'(u)du\right)\frac{\partial}{\partial s}\{1-\vartheta_\lambda^\alpha(x,s)\}\,ds\right|\\
&=\left|\left(\int_x^z f_x'(u)du\right)\{1-\vartheta_\lambda^\alpha(x,z)\}-\int_x^z f_x'(s)\{1-\vartheta_\lambda^\alpha(x,s)\}\,ds\right|\\
&\quad-\left|\left(\int_x^z f_x'(u)du\right)\{1-\vartheta_\lambda^\alpha(x,z)\}-\int_z^1 f_x'(s)\{1-\vartheta_\lambda^\alpha(x,s)\}\,ds\right|\\
&=\left|\int_x^z f_x'(s)\{1-\vartheta_\lambda^\alpha(x,s)\}\,ds+\int_z^1 f_x'(s)\{1-\vartheta_\lambda^\alpha(x,s)\}\,ds\right|\\
&\leq\int_x^z\bigvee_x^s f_x'ds+\frac{C\alpha x(1-x)}{\lambda}\int_z^1\left(\bigvee_x^s f_x'\right)(s-x)^{-2}ds.
\end{aligned}$$

Now, let $z=x+\frac{1-x}{\sqrt{\lambda}}$ and then putting $u=\frac{1-x}{s-x}$, we get

$$\begin{aligned}
\left|\int_x^1\left(\int_x^s f_x'(u)du\right)\frac{\partial}{\partial s}\{\mathcal{M}_\lambda^\alpha(x,s)\}\,ds\right|&\leq\frac{1-x}{\sqrt{\lambda}}\left(\bigvee_x^{x+\frac{1-x}{\sqrt{\lambda}}}f_x'\right)+\frac{C\alpha x(1-x)}{\lambda}\int_{x+\frac{1-x}{\sqrt{\lambda}}}^1\left(\bigvee_x^s f_x'\right)(s-x)^{-2}ds\\
&\leq\frac{1-x}{\sqrt{\lambda}}\left(\bigvee_x^{x+\frac{1-x}{\sqrt{\lambda}}}f_x'\right)+\frac{C\alpha x(1-x)}{\lambda}\int_1^{\sqrt{\lambda}}\left(\bigvee_x^{x+\frac{1-x}{u}}f_x'\right)(1-x)^{-1}du\\
&\leq\frac{1-x}{\sqrt{\lambda}}\left(\bigvee_x^{x+\frac{1-x}{\sqrt{\lambda}}}f_x'\right)+\frac{C\alpha x}{\lambda}\sum_{k=1}^{[\sqrt{\lambda}]}\left(\bigvee_x^{x+\frac{1-x}{k}}f_x'\right).
\end{aligned}\tag{2.31}$$

Collecting estimates from (2.29-2.31), we get required result.

2.1.8 Bivariate Generalization of Ismail-May Operators

In this section, we introduce the bivariate generalization of the operators (2.4). A lot of work has already been done on constructing the bivariate form of various positive linear operators and analyzing their convergence results. We would like to recommend to the readers some interesting articles [19; 22; 60; 61; 70; 89; 103; 116; 140; 149; 155] for more information. The bivariate extension of the operators (2.4) for $(x,y)\in I^2=[0,1]\times[0,1]$ and $\lambda_1>0,\lambda_2>0$ is defined as follows:

$$R_{\lambda_1\lambda_2}^{k_1k_2}(f;x,y)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\hbar_{\lambda_1\lambda_2}^{k_1k_2}(x,y)f\Big(\frac{k_1}{\lambda_1+k_1},\frac{k_2}{\lambda_2+k_2}\Big),\tag{2.32}$$

where the basis function is considered as:

$$\hbar_{\lambda_1\lambda_2}^{k_1k_2}(x,y) = e^{-(\lambda_1 x+\lambda_2 y)}\frac{\lambda_1\lambda_2(\lambda_1+k_1)^{k_1}(\lambda_2+k_2)^{k_2}}{k_1!k_2!}(xe^{-x})^{k_1}(ye^{-y})^{k_2}.$$

Lemma 2.1.16 *Let* $e_{n_1n_2}(s_1,s_2) = s_1^{n_1}s_2^{n_2}, 0 \le n_1+n_2 \le 2$*. For* $(x,y) \in I^2 = [0,1]\times[0,1]$ *and* $\lambda_1,\lambda_2 \in [0,\infty)$*, we have*

$$\begin{aligned}
(R_\lambda e_{00}(s_1,s_2))(x,y) &= 1;\\
(R_\lambda e_{10}(s_1,s_2))(x,y) &= \frac{\lambda_1}{\lambda_1+1}x;\\
(R_\lambda e_{01}(s_1,s_2))(x,y) &= \frac{\lambda_2}{\lambda_2+1}y;\\
(R_\lambda e_{20}(s_1,s_2))(x,y) &= \frac{{\lambda_1}^2}{(\lambda_1+1)(\lambda_1+2)}x^2 + \frac{\lambda_1}{(\lambda_1+1)^2}x;\\
(R_\lambda e_{02}(s_1,s_2))(x,y) &= \frac{{\lambda_2}^2}{(\lambda_2+1)(\lambda_2+2)}y^2 + \frac{\lambda_2}{(\lambda_2+1)^2}y.
\end{aligned}$$

Remark 2.1.17 *Using Lemma 2.1.16, we have*

$$\begin{aligned}
(R_{k_1k_2}^{\lambda_1\lambda_2}(e_{10}-x))(x,y) &= -\frac{x}{(\lambda_1+1)};\\
(R_{k_1k_2}^{\lambda_1\lambda_2}(e_{01}-y))(x,y) &= -\frac{y}{(\lambda_2+1)};\\
(R_{\lambda_1\lambda_2}^{k_1k_2}(e_{20}-x)^2)(x,y) &= -\frac{(\lambda_1-2)}{(\lambda_1+1)(\lambda_1+2)}x^2 + \frac{\lambda_1}{(\lambda_1+2)}x;\\
(R_{\lambda_1\lambda_2}^{k_1k_2}(e_{02}-y)^2)(x,y) &= -\frac{(\lambda_2-2)}{(\lambda_2+1)(\lambda_2+2)}y^2 + \frac{\lambda_2}{(\lambda_2+2)}y.
\end{aligned}$$

Now we estimate the degree of approximation of bivariate operators (2.32) with the help of Lipschitz class functions. We define Lipschitz class $Lip_M(\zeta_1,\zeta_2)$ for bivariate functions for $0<\zeta_1\le 1$ and $0<\zeta_2\le 1$ as follows:

$$|f(s_1,s_2)-f(x,y)| \le M|s_1-x|^{\zeta_1}|s_2-y|^{\zeta_2}.$$

Theorem 2.1.18 *If* $f \in Lip_M(\zeta_1,\zeta_2)$*, then for* $\zeta_1,\zeta_2 \in (0,1]$

$$\left|R_{\lambda_1\lambda_2}^{k_1k_2}(f;x,y) - f(x,y)\right| \le M\delta_n^{\frac{\zeta_1}{2}}\delta_m^{\frac{\zeta_2}{2}}.$$

Proof: If $f \in Lip_M(\zeta_1,\zeta_2)$, we can write

$$\begin{aligned}
&\left|R_{\lambda_1\lambda_2}^{k_1k_2}(f;x,y) - f(x,y)\right|\\
&\le \left(R_{\lambda_1\lambda_2}^{k_1k_2}|f(s_1,s_2)-f(x,y)|\right)(x,y)\\
&\le \left(R_{\lambda_1\lambda_2}^{k_1k_2}\left(M|s_1-x|^{\zeta_1}|s_2-y|^{\zeta_2}\right)\right)(x,y)\\
&\le M\left(R_{\lambda_1}|s_1-x|^{\zeta_1}\right)(x)\left(R_{\lambda_2}|s_2-y|^{\zeta_2}\right)(y).
\end{aligned}$$

Using Hölder's inequality

$$
\begin{aligned}
&\left|R_{\lambda_1\lambda_2}^{k_1k_2}(f;x,y)-f(x,y)\right|\\
&\leq M\Big(R_{\lambda_1}(e_{10}-x)^2\Big)^{\frac{\zeta_1}{2}}(x,y)\left(R_{\lambda_1}(e_{00})\right)^{\frac{2-\zeta_1}{2}}(x,y)\\
&\times\Big(R_{\lambda_2}(e_{01}-y)^2\Big)^{\frac{\zeta_2}{2}}(x,y)\left(R_{\lambda_2}(e_{00})\right)^{\frac{2-\zeta_2}{2}}(x,y)\\
&\leq M\delta_n^{\frac{\zeta_1}{2}}\delta_m^{\frac{\zeta_2}{2}}.
\end{aligned}
$$

2.2 On modification of certain exponential type operators preserving constant and e^{-x}

2.2.1 Introduction

In the year 2003, King achieved recognition for his work on modified Bernstein operators which preserve test functions e_0 and e_2 on $[0, 1]$. As a result, King's research sparked the interest of many researchers in this particular area and they put forward many relevant studies. Numerous researchers have made significant contributions in this direction by defining operators that preserve e_0 and e_2, $e_2 + ae_1$ for $a > 0$, linear functions, exponential functions etc. Depending on the ultimate goal of this research, the scope of this study will be limited to a smaller scale, with a specific focus on the preservation of exponential functions exclusively. To the best of our knowledge, the study of preservation of exponential functions is currently in its early stage. Here we represent some most recent references which are relevant to this study.

In their paper published in 2017, Acar et al. [6] introduced a modified version of the Szász-Mirakjan operators that exhibit the property of preserving the function e^{2ax}, where a is a positive constant. They presented a comparative analysis between modified operators and the Szász-Mirakjan operators and examined their respective shape preservation properties. The error was also estimated by using a natural transformation in terms of the first-order modulus of continuity. Aral et al. [30] expanded the study of these modified operators [6] and demonstrated the usefulness of these operators from a computational point of view. In their study, Acar et al. [5] considered Szász-Mirakjan operators. These operators were designed to simultaneously fix exponential functions of the form e^{ax} and e^{2ax} with $a > 0$. Additionally, the authors defined a new weighted modulus of smoothness in order to establish the approximation order. Additionally, the researchers provided saturation results as a means of validating the accuracy of the estimates for the modified operators.

Over the past four years, numerous researchers have made modifications to various operators, such as the Bernstein operators [27], the Stancu type Szász-Mirakjan-Durrmeyer operators [113], the Baskakov-Szász-Mirakjan operators [88], the Baskakov-Schurer-Szász-Stancu operators [150], the Baskakov-Schurer-Szász operators [171], and the Phillips operators [92; 164]. Deo et al. [62; 65] introduced a sequence of operators based on King's approach that provides a better rate of convergence compared to the Szász-Mirakjan Durrmeyer and Baskakov Durrmeyer operators. In 2018, Yilmaz et al. [172] made modifications to the Baskakov-Kantorovich operators, resulting in the development of a sequence of operators that preserves the function e^{-x} and constant functions [172]. The overall goal of this article is to propose the modification of the operators [102] while maintaining the preservation of the functions e^{-x} and constant functions.

2.2.2 Construction of the Operators

May [129] has done excellent work by defining exponential operators $\mathcal{L}_n$ as

$$\mathcal{L}_n(f;x) = \int_{-\infty}^{\infty} \mathcal{W}(n,x,t) f(t) dt,$$

where $\mathcal{W}$ is the kernel which satisfies two conditions given as follows:

1. $\mathcal{L}_n(1;x) = 1$ normalisation condition.
2. $\frac{\partial}{\partial x}\mathcal{W}(n,x,t) = \frac{(t-x)n}{p(x)}\mathcal{W}(n,x,t)$, where $p(x)$ is analytic and positive for $x \in (-\infty, \infty)$.

This work was carried forward by Ismail-May [102]. They considered a couple of more exponential operators and investigated their convergence properties. Using above definition, the authors reintroduced certain well-known operators, such as Bernstein operators, Szász operators etc. and constructed some new operators which were later studied in [86; 123; 134]. Among these new operators defined in [102], one operator is given as:

$$\mathcal{T}_n(f;x) = e^{-n\sqrt{x}} \left\{ f(0) + n \int_0^{\infty} e^{-nt/\sqrt{x}} t^{-1/2} I_1\left(2n\sqrt{t}\right) f(t)\, dt \right\}, \tag{2.33}$$

where I_1 is modified Bessel's function of first kind defined as

$$I_m(z) = \sum_{j=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{m+2j}}{j!\Gamma(m+j+1)}.$$

In a study conducted by Gupta [85], the moments and central moments were calculated for the operators described in (2.33). Additionally, convergence estimates and direct results were obtained.

The objective of this article is to construct operators that preserve e_0 and e^{-x}. It is assumed that operators (2.33) maintain the preservation of e^{-x}, then

$$\begin{aligned}
\mathcal{T}_n\left(e^{-t};x\right) &= e^{-n\sqrt{\varrho_n(x)}}\left\{1+n\int_0^\infty e^{-nt/\sqrt{\varrho_n(x)}}t^{-1/2}\sum_{j=0}^\infty \frac{\left(n\sqrt{t}\right)^{1+2j}}{j!\Gamma(j+2)}e^{-t}dt\right\}\\
&= e^{-n\sqrt{\varrho_n(x)}}\left\{1+\sum_{j=0}^\infty \frac{n^{2(1+j)}}{j!\Gamma(j+2)}\int_0^\infty e^{-\left(\frac{nt+\sqrt{\varrho_n(x)}t}{\sqrt{\varrho_n(x)}}\right)}t^j dt\right\}\\
&= e^{-n\sqrt{\varrho_n(x)}}\left\{\sum_{j=0}^\infty \frac{n^{2(1+j)}}{j!\Gamma(j+2)}\left(\frac{\sqrt{\varrho_n(x)}}{n+\sqrt{\varrho_n(x)}}\right)^{j+1}\int_0^\infty e^{-u}u^j du+1\right\}\\
&= e^{-n\sqrt{\varrho_n(x)}}\left\{1+\sum_{j=0}^\infty \frac{n^{2(1+j)}}{(j+1)!}\left(\frac{\sqrt{\varrho_n(x)}}{n+\sqrt{\varrho_n(x)}}\right)^{j+1}\right\}\\
&= e^{-n\sqrt{\varrho_n(x)}}\left\{\sum_{j=0}^\infty \frac{1}{j!}\left(\frac{n^2\sqrt{\varrho_n(x)}}{n+\sqrt{\varrho_n(x)}}\right)^{j}\right\}\\
&= e^{\frac{-n\varrho_n(x)}{n+\sqrt{\varrho_n(x)}}}.
\end{aligned}$$

Taking into account $\mathcal{T}_n\left(e^{-t};x\right)=e^{-x}$, then we can find without hesitation

$$\varrho_n(x)=\frac{x^{3/2}\sqrt{4n^2+x}+2n^2x+x^2}{2n^2}. \tag{2.34}$$

For $x\in[0,\infty)$, we consider the following modified form of operators (2.33)

$$\ddot{\mathcal{T}}_n(f;x)=e^{-n\sqrt{\varrho_n(x)}}\left\{f(0)+n\int_0^\infty e^{-nt/\sqrt{\varrho_n(x)}}t^{-1/2}I_1\left(2n\sqrt{t}\right)f(t)\,dt\right\}, \tag{2.35}$$

where $\varrho_n(x)$ is given as above.

2.2.3 Preliminaries

After simple calculations, the mgf of the operators (2.35) may be given as

$$\ddot{\mathcal{T}}_n\left(e^{\phi t};x\right)=e^{\frac{n\phi\varrho_n(x)}{n-\phi\sqrt{\varrho_n(x)}}}. \tag{2.36}$$

Since the moments are related with the mgf, the m-th moment $\ddot{\mathcal{T}}_n(e_m;x)$, $e_m(t)=t^m$ $(m\in\mathbb{N}\cup\{0\})$ may be obtained by the following relation:

$$\ddot{\mathcal{T}}_n(e_m;x)=\left[\frac{\partial^m}{\partial\phi^m}\ddot{\mathcal{T}}_n\left(e^{\phi t};x\right)\right]_{\phi=0}=\left[\frac{\partial^m}{\partial\phi^m}\left(e^{\frac{n\phi\varrho_n(x)}{n-\phi\sqrt{\varrho_n(x)}}}\right)\right]_{\phi=0}.$$

Employing Mathematica, the expansion of above expression in powers of ϕ may be given as:

$$\begin{aligned}
&\ddot{\mathcal{T}}_n\left(e^{\phi t};x\right)\\
&= e^{\frac{n\phi\varrho_n(x)}{n-\phi\sqrt{\varrho_n(x)}}}\\
&= 1+\phi\varrho_n(x)+\phi^2\left(\frac{\varrho_n(x)^{3/2}}{n}+\frac{\varrho_n(x)^2}{2}\right)+\phi^3\left(\frac{\varrho_n(x)^2}{n^2}+\frac{\varrho_n(x)^{5/2}}{n}+\frac{\varrho_n(x)^3}{6}\right)\\
&+\frac{\phi^4\left(n^3\varrho_n(x)^4+12n^2\varrho_n(x)^{7/2}+36n\varrho_n(x)^3+24\varrho_n(x)^{5/2}\right)}{24n^3}+O\left(\phi^5\right).
\end{aligned}$$

Also, by change of scale property of mgf, if we expand $e^{-\phi x}\ddot{\mathcal{T}}_n\left(e^{\phi t};x\right)$ in powers of ϕ, the central moment of m-th order $\nu_m(x)=\ddot{\mathcal{T}}_n\left((t-x)^m;x\right)$ can be obtained by collecting the coefficient of $\frac{\phi^m}{m!}$.

$$\begin{aligned}
&e^{-\phi x}\ddot{\mathcal{T}}_n\left(e^{\phi t};x\right)\\
&= e^{-\phi x+\frac{n\phi\varrho_n(x)}{n-\phi\sqrt{\varrho_n(x)}}}\\
&= 1+\phi(\varrho_n(x)-x)+\phi^2\left(\frac{\varrho_n(x)^{3/2}}{n}+\frac{x^2}{2}-x\varrho_n(x)+\frac{\varrho_n(x)^2}{2}\right)\\
&+\phi^3\left(\frac{\varrho_n(x)^2}{n^2}-\frac{x\varrho_n(x)^{3/2}}{n}+\frac{\varrho_n(x)^{5/2}}{n}-\frac{x^3}{6}+\frac{x^2\varrho_n(x)}{2}-\frac{x\varrho_n(x)^2}{2}+\frac{\varrho_n(x)^3}{6}\right)\\
&+\phi^4\frac{\left[\begin{array}{l} n^3x^4-4n^3x^3\varrho_n(x)+6n^3x^2\varrho_n(x)^2-4n^3x\varrho_n(x)^3+n^3\varrho_n(x)^4+12n^2x^2\varrho_n(x)^{3/2}\\ -24n^2x\varrho_n(x)^{5/2}+12n^2\varrho_n(x)^{7/2}-24nx\varrho_n(x)^2+36n\varrho_n(x)^3+24\varrho_n(x)^{5/2}\end{array}\right]}{24n^3}\\
&+O\left(\phi^5\right).
\end{aligned}\tag{2.37}$$

Lemma 2.2.1 *Following the above argument, we can find the first four moments as follows:*

$$\begin{aligned}
\ddot{\mathcal{T}}_n(e_0;x) &= 1;\\
\ddot{\mathcal{T}}_n(e_1;x) &= \varrho_n(x);\\
\ddot{\mathcal{T}}_n(e_2;x) &= (\varrho_n(x))^2+\frac{2(\varrho_n(x))^{3/2}}{n};\\
\ddot{\mathcal{T}}_n(e_3;x) &= (\varrho_n(x))^3+\frac{6(\varrho_n(x))^{5/2}}{n}+\frac{6(\varrho_n(x))^2}{n^2};\\
\ddot{\mathcal{T}}_n(e_4;x) &= (\varrho_n(x))^4+\frac{12(\varrho_n(x))^{7/2}}{n}+\frac{36(\varrho_n(x))^3}{n^2}+\frac{24(\varrho_n(x))^{5/2}}{n^3}.
\end{aligned}$$

Lemma 2.2.2 *Using (2.37), we have the central moments of the modified operator (2.35) as:*

$$
\begin{aligned}
\nu_1(x) &= \varrho_n(x) - x;\\
\nu_2(x) &= (\varrho_n(x) - x)^2 + \frac{2(\varrho_n(x))^{3/2}}{n};\\
\nu_3(x) &= (\varrho_n(x))^3 + \frac{6(\varrho_n(x))^{5/2}}{n} + \left(\frac{6}{n^2} - 3x\right)(\varrho_n(x))^2\\
&\quad - \frac{6x(\varrho_n(x))^{3/2}}{n} + 3x^2\varrho_n(x) - x^3;\\
\nu_4(x) &= (\varrho_n(x))^4 + \frac{12(\varrho_n(x))^{7/2}}{n} + \left(\frac{36}{n^2} - 4x\right)(\varrho_n(x))^3 + \left(\frac{24}{n^3} - \frac{24x}{n}\right)(\varrho_n(x))^{5/2}\\
&\quad + \left(\frac{6}{n^2} + 6x^2\right)(\varrho_n(x))^2 + \frac{12x^2(\varrho_n(x))^{3/2}}{n} - 4x^3\varrho_n(x) + x^4.
\end{aligned}
$$

Also,

$$\lim_{n\to\infty} n\nu_1(x) = \lim_{n\to\infty} n\left[\varrho_n(x) - x\right] = x^{3/2}$$

and

$$\lim_{n\to\infty} n\nu_2(x) = \lim_{n\to\infty} n\left[(\varrho_n(x) - x)^2 + \frac{2(\varrho_n(x))^{3/2}}{n}\right] = 2x^{3/2}.$$

2.2.4 Main Results

Let us represent the subspace of real-valued continuous functions having finite limit at infinity equipped with uniform norm by $C^*[0,\infty)$. In 1970, Boyanov [44] conducted a study on the approximation properties of a function defined on an infinite interval. Later, Holhoş [100] verified the next theorem in order to quantify the rate of convergence of a function.

Theorem 2.2.3 *Let $\mathcal{L}_n : C^*[0,\infty) \to C^*[0,\infty)$ be the sequence of positive linear operators and*

$$
\begin{aligned}
\|\mathcal{L}_n(e_0) - 1\|_{[0,\infty)} &= \beta_n,\\
\left\|\mathcal{L}_n(e^{-t}) - e^{-x}\right\|_{[0,\infty)} &= \gamma_n,\\
\left\|\mathcal{L}_n\left(e^{-2t}\right) - e^{-2x}\right\|_{[0,\infty)} &= \delta_n.
\end{aligned}
$$

Then

$$\|\mathcal{L}_n f - f\|_{[0,\infty)} \le \beta_n\|f\|_{[0,\infty)} + (2+\beta_n)\omega^*(f;\sqrt{\beta_n + 2\gamma_n + \delta_n}).$$

The modulus of continuity is defined as:

$$\omega^*(\hbar;\delta) = \sup_{\substack{|e^{-t}-e^{-x}|\le\delta\\ x,t>0}} |\hbar(t) - \hbar(x)|$$

with the property

$$|\hbar(t) - \hbar(x)| \leq \left(1 + \frac{(e^{-t} - e^{-x})^2}{\delta^2}\right)\omega^*(\hbar;\delta), \quad \delta > 0. \tag{2.38}$$

In next theorem we give quantitative estimate for proposed operators as an application of above mentioned theorem.

Theorem 2.2.4 *For $f \in C^*[0,\infty)$, we have*

$$\left\|\ddot{\mathcal{T}}_n f - f\right\|_{[0,\infty)} \leq 2\omega^*\left(f;\ \sqrt{\delta_n}\right).$$

Here $\ddot{\mathcal{T}}_n f$ converges to f uniformly and $\delta_n \to 0$ as $n \to \infty$.

Proof: The operators preserve e^{-x} as well as constant functions so $\beta_n = \gamma_n = 0$. we only have to evaluate δ_n. From (2.36), we have

$$\ddot{\mathcal{T}}_n\left(e^{-2t};x\right) = e^{\frac{-2n\varrho_n(x)}{n+2\sqrt{\varrho_n(x)}}}$$

where

$$\varrho_n(x) = \frac{2n^2x + x^2 + x^{3/2}\sqrt{4n^2+x}}{2n^2}.$$

Using mathematica, we will get

$$\ddot{\mathcal{T}}_n\left(e^{-2t};x\right) = e^{-2x} + \frac{\left(2e^{-2x}\right)x^{3/2}}{n} + \frac{\left(e^{-2x}x^2\right)(2x-3)}{n^2} + O\left(\left(\frac{1}{n}\right)^3\right).$$

Since

$$\sup_{x\in[0,\infty)} x^{3/2}e^{-2x} = \frac{3\sqrt{3}}{8e^{3/2}},\ \sup_{x\in[0,\infty)} x^2e^{-2x} = \frac{1}{e^2}$$

and

$$\sup_{x\in[0,\infty)} x^3e^{-2x} = \frac{27}{8e^3}.$$

So, we get

$$\begin{aligned}
\delta_n &= \left\|\ddot{\mathcal{T}}_n\left(e^{-2t}\right) - e^{-2x}\right\|_{[0,\infty)} \\
&= \sup_{x\in[0,\infty)}\left|\ddot{\mathcal{T}}_n\left(e^{-2t}\right) - e^{-2x}\right| \\
&\leq \frac{1}{n}\left(\frac{3\sqrt{3}}{4e^{3/2}}\right) + \frac{1}{n^2}\left(\frac{3}{e^2} + \frac{27}{4e^3}\right) + O\left(\left(\frac{1}{n}\right)^3\right) \\
&\leq O\left(\frac{1}{n}\right) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Remark 2.2.5 *Using Mathematica and Lemma 2.2.2, we have*

$$\begin{aligned}\lim_{n\to\infty} n^2 \nu_4(x) &= \lim_{n\to\infty} n^2 \Bigg[(\varrho_n(x))^4 + \frac{12(\varrho_n(x))^{7/2}}{n} + \left(\frac{36}{n^2} - 4x\right)(\varrho_n(x))^3 \\ &\quad + \left(\frac{24}{n^3} - \frac{24x}{n}\right)(\varrho_n(x))^{5/2} + \left(\frac{6}{n^2} + 6x^2\right)(\varrho_n(x))^2 + \frac{12x^2(\varrho_n(x))^{3/2}}{n} - 4x^3\varrho_n(x) + x^4 \Bigg] \\ &= 12x^3\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} n^2 \ddot{\mathcal{T}}_n\left((e^{-x} - e^{-t})^4; x\right) &= \lim_{n\to\infty} n^2 \ddot{\mathcal{T}}_n\left(\sum_{j=0}^{4} \binom{4}{j} (e^{-x})^j (e^{-t})^{4-j}\right) \\ &= \lim_{n\to\infty} n^2 \sum_{j=0}^{4} \binom{4}{j} e^{-jx} \ddot{\mathcal{T}}_n\left(e^{-(4-j)t}; x\right) \\ &= \lim_{n\to\infty} n^2 \sum_{j=0}^{4} \binom{4}{j} e^{-jx} e^{\frac{-(4-j)n\varrho_n(x)}{\left(n+(4-j)\sqrt{\varrho_n(x)}\right)}} \\ &= 12e^{-4x}x^3.\end{aligned}$$

Theorem 2.2.6 *For $x \in [0, \infty)$, and $f, f'' \in C^*[0, \infty)$ we have*

$$\begin{aligned}&\left| n\left[\ddot{\mathcal{T}}_n(f; x) - f(x)\right] - x^{3/2}\left[f'(x) + f''(x)\right] \right| \\ &\le |a_n(x)|\,|f'(x)| + |b_n(x)|\,|f''(x)| + 2\omega^*(f''; \delta)\left(\left(2b_n(x) + x^{3/2}\right) + c_n(x)\right).\end{aligned}$$

Proof: By Taylor's expansion we have

$$f(t) = f(x) + (t - x) f'(x) + \frac{1}{2}(t - x)^2 f''(x) + \ddot{r}(t, x)(t - x)^2$$

where

$$\ddot{r}(t, x) = \frac{f''(\mu) - f''(x)}{2}, \quad x < \mu < t.$$

From Lemma 2.2.2 and applying $\ddot{\mathcal{T}}_n$ to both sides of the above expression, we have

$$\left| \ddot{\mathcal{T}}_n(f; x) - f(x) - \nu_1(x) f'(x) - \frac{1}{2}\nu_2(x) f''(x) \right| \le \left| \ddot{\mathcal{T}}_n\left(\ddot{r}(t, x)(t - x)^2; x\right) \right|.$$

Using Lemma 2.2.2

$$\begin{aligned}&\left| n\left[\ddot{\mathcal{T}}_n(f; x) - f(x)\right] - \left(x^{3/2}\right) f'(x) - \frac{1}{2}\left(2x^{3/2}\right) f''(x) \right| \\ &\le \left| n(\nu_1(x)) - \left(x^{3/2}\right) \right| |f'(x)| + \frac{1}{2}\left| n(\nu_2(x)) - \left(2x^{3/2}\right) \right| |f''(x)| \\ &+ \left| n\ddot{\mathcal{T}}_n\left(\ddot{r}(t, x)(t - x)^2; x\right) \right|.\end{aligned}$$

Taking $a_n(x) = n(\nu_1(x)) - \left(x^{3/2}\right)$ and $b_n(x) = \frac{1}{2}\left|n(\nu_2(x)) - \left(2x^{3/2}\right)\right|$, we get

$$\begin{aligned}&\left|n\left[\ddot{\mathcal{T}}_n(f;x) - f(x)\right] - \left(x^{3/2}\right)[f'(x) + f''(x)]\right|\\ &\leq |a_n(x)||f'(x)| + |b_n(x)||f''(x)| + \left|n\ddot{\mathcal{T}}_n\left(\ddot{r}(t,x)(t-x)^2;x\right)\right|.\end{aligned}$$

For completion of the proof, we need to evaluate $\left|n\ddot{\mathcal{T}}_n\left(\ddot{r}(t,x)(t-x)^2;x\right)\right|$. Applying inequality (2.38), we get

$$|\ddot{r}(t,x)| \leq \left(1 + \frac{\left(e^{-t}-e^{-x}\right)^2}{\delta^2}\right)\omega^*(f'';\delta).$$

Two inequality $|\ddot{r}(t,x)| \leq 2\omega^*(f'';\delta)$ and $|\ddot{r}(t,x)| \leq \frac{2\left(e^{-t}-e^{-x}\right)^2}{\delta^2}\omega^*(f'';\delta)$ holds for the case $\left|e^{-t}-e^{-x}\right| \leq \delta$ and $\left|e^{-t}-e^{-x}\right| > \delta$ respectively.
Thus

$$|\ddot{r}(t,x)| \leq 2\left(1 + \frac{\left(e^{-t}-e^{-x}\right)^2}{\delta^2}\omega^*(f'';\delta)\right).$$

Using above argument and Cauchy Schwarz inequality, we get

$$\begin{aligned}&n\ddot{\mathcal{T}}_n\left(\ddot{r}(t,x)(t-x)^2;x\right)\\ &\leq n\ddot{\mathcal{T}}_n\left(2\left(1 + \frac{\left(e^{-t}-e^{-x}\right)^2}{\delta^2}\omega^*(f'';\delta)\right)(t-x)^2;x\right)\\ &= 2n(\nu_2(x))\,\omega^*(f'';\delta) + \frac{2n}{\delta^2}\omega^*(f'';\delta)\,\ddot{\mathcal{T}}_n\left(\left(e^{-t}-e^{-x}\right)^2(t-x)^2;x\right)\\ &= 2\omega^*(f'';\delta)\left[n(\nu_2(x)) + \left(n^2\nu_4(x)\right)^{1/2}\left(n^2\ddot{\mathcal{T}}_n\left(\left(e^{-t}-e^{-x}\right)^2;x\right)\right)^{1/2}\right].\end{aligned}$$

We complete the proof by choosing $\delta = \frac{1}{\sqrt{n}}$ and

$$c_n(x) = \left(n^2\nu_4(x)\right)^{1/2}\left(n^2\ddot{\mathcal{T}}_n\left(\left(e^{-t}-e^{-x}\right)^2;x\right)\right)^{1/2}.$$

Theorem 2.2.7 *Let* $x \in [0,\infty)$ *and* $f, f'' \in C^*[0,\infty)$. *Then we have*

$$\lim_{n\to\infty} n\left[\ddot{\mathcal{T}}_n(f;x) - f(x)\right] = x^{3/2}[f'(x) + f''(x)]. \tag{2.39}$$

Proof: By the Taylor's expansion of f, we have

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2}(t-x)^2 f''(x) + \ddot{r}(t,x)(t-x)^2 \tag{2.40}$$

where

$$\lim_{t\to x}\ddot{r}(t,x) = 0.$$

From Lemma 2.2.2 and applying $\ddot{T}_n$ to (2.40), we get

$$\ddot{\mathcal{T}}_n(f;x) - f(x) = \nu_1(x)f'(x) + \frac{1}{2}\nu_2(x)f''(x) + \ddot{\mathcal{T}}_n\left(\ddot{r}(t,x)(t-x)^2;x\right).$$

Making use of Cauchy Schwarz inequality, we have

$$\ddot{\mathcal{T}}_n\left(\ddot{r}(t,x)(t-x)^2;x\right) \leq \sqrt{\ddot{\mathcal{T}}_n\left(\ddot{r}^2(t,x);x\right)\ddot{\mathcal{T}}_n\left((t-x)^4;x\right)}. \tag{2.41}$$

Also, we have

$$\lim_{n\to\infty}\ddot{\mathcal{T}}_n\left(\ddot{r}^2(t,x);x\right) = 0. \tag{2.42}$$

From (2.41) and (2.42), we get

$$\lim_{n\to\infty} n\ddot{\mathcal{T}}_n\left(\ddot{r}(t,x)(t-x)^2;x\right) = 0.$$

Thus, we get

$$\begin{aligned}
&\lim_{n\to\infty} n\left[\ddot{\mathcal{T}}_n(f;x) - f(x)\right] \\
&= \lim_{n\to\infty} n\left[\nu_1(x) f'(x) + \frac{1}{2}\nu_2(x) f''(x) + \ddot{\mathcal{T}}_n\left(\ddot{r}(t,x)(t-x)^2;x\right)\right] \\
&= x^{3/2}\left[f'(x) + f''(x)\right].
\end{aligned}$$

Let us represent the class of bounded and uniform continuous functions on $[0,\infty)$ equipped with sup norm by $C_B[0,\infty)$. Our subsequent theorems determine the degree of approximation for our proposed operators $\ddot{\mathcal{T}}_n$ in terms of usual and second order modulus of continuity.

Theorem 2.2.8 *Let* $f \in C_B[0,\infty)$. *Then, for all* $x \in [0,\infty)$, *there exists a positive constant* $\mathcal{M}$ *such that*

$$\left|\ddot{\mathcal{T}}_n(f;x) - f(x)\right| \leq \mathcal{M}\omega_2\left(f;\frac{1}{2}\sqrt{\nu_2(x) + \frac{(\varrho_n(x)-x)^2}{2}}\right) + \omega\left(f;(\varrho_n(x)-x)\right).$$

Proof: We construct the auxiliary operators $\mathbb{T}_n : C_B[0,\infty) \to C_B[0,\infty)$

$$\mathbb{T}_n(f;x) = \ddot{\mathcal{T}}_n(f;x) + f(x) - f(\varrho_n(x)),$$

where $\varrho_n(x)$ is given in 2.34.
For the operators 2.35, we have

$$\left\|\ddot{\mathcal{T}}_n(f;x)\right\| \leq \|f\|$$

implies

$$\|\mathbb{T}_n(f;x)\| \leq \left\|\ddot{\mathcal{T}}_n(f;x)\right\| + 2\|f\| \leq 3\|f\|. \tag{2.43}$$

Also the Taylor expansion for $\hbar \in C_B^2[0,\infty)$, is given as

$$\hbar(t) = \hbar(x) + (t-x)\hbar'(x) + \int_x^t (t-\mu)\hbar''(\mu)\,d\mu, x \in [0,\infty).$$

Applying Cauchy schwarz inequality and $\mathbb{T}_n$ to both sides of the above equation, we get

$$\begin{aligned} |\mathbb{T}_n(\hbar;x) - \hbar(x)| &= \left|\mathbb{T}_n\left(\int_x^t (t-\mu)\hbar''(\mu)\,d\mu; x\right)\right| \\ &\leq \left|\ddot{\mathcal{T}}_n\left(\int_x^t (t-\mu)\hbar''(\mu)\,d\mu; x\right)\right| + \left|\int_x^{\varrho_n(x)} (\varrho_n(x)-\mu)\hbar''(\mu)\,d\mu\right| \\ &\leq \|\hbar''\|\left(\nu_2(x) + \frac{(\varrho_n(x)-x)^2}{2}\right). \end{aligned} \quad (2.44)$$

Using the estimates from equation (2.43) and (2.44), we get

$$\begin{aligned} \left|\ddot{\mathcal{T}}_n(f;x) - f(x)\right| &\leq |\mathbb{T}_n(f-\hbar;x) - (f-\hbar)(x)| + |f(\varrho_n(x)) - f(x)| + |\mathbb{T}_n(\hbar;x) - \hbar(x)| \\ &\leq 4\|f-\hbar\| + \left(\nu_2(x) + \frac{(\varrho_n(x)-x)^2}{2}\right)\|\hbar''\| + |f(\varrho_n(x)) - f(x)| \\ &\leq 4K_2\left(f; \frac{1}{4}\left(\nu_2(x) + \frac{(\varrho_n(x)-x)^2}{2}\right)\right) + |f(\varrho_n(x)) - f(x)| \\ &\leq \mathcal{M}\omega_2\left(f; \frac{1}{2}\sqrt{\left(\nu_2(x) + \frac{(\varrho_n(x)-x)^2}{2}\right)}\right) + \omega(f; (\varrho_n(x)-x)). \end{aligned}$$

2.2.5 Comparison with T_n

In the following theorem, we demonstrate that the newly constructed operators $\ddot{\mathcal{T}}_n$ provide improved approximation compared to the original operators $\mathcal{T}_n$ for a particular class of functions. This is achieved by utilizing the asymptotic formulae satisfied by $\mathcal{T}_n$ and $\ddot{\mathcal{T}}_n$.

Theorem 2.2.9 *Let $f \in C^2[0,\infty)$. Assume that there exist $n_0 \in \mathbb{N}$, such that*

$$f(x) \leq \ddot{\mathcal{T}}_n(f;x) \leq \mathcal{T}(f;x) \quad \forall n \geq n_0, \quad x \in (0,\infty) \quad (2.45)$$

then

$$f''(x) \geq -f'(x) \geq 0, x \in (0,\infty). \quad (2.46)$$

In particular $f''(x) \geq 0$.

Contrarily, if (2.46) holds with strict inequalities for a given $x \in (0, \infty)$, there exist $n_0 \in \mathbb{N}$, such that for $n \geq n_0$

$$f(x) < \ddot{\mathcal{T}}_n(f; x) < \mathcal{T}(f; x). \tag{2.47}$$

Proof: From (2.45) we have

$$0 \leq n(\ddot{\mathcal{T}}_n(f; x) - f(x)) \leq n(\mathcal{T}(f; x) - f(x)) \quad \forall n \geq n_0, \quad x \in (0, \infty).$$

Considering an asymptotic formula which is held by operators $\mathcal{T}_n$ defined in [85].

$$\lim_{n\to\infty} n(\mathcal{T}(f; x) - f(x)) = x^{3/2} f''(x).$$

Now considering (2.39) and above equation, we get

$$0 \leq -f'(x) \leq f''(x).$$

Contrarily, if (2.46) holds with strict inequality for a given $x \in (0, \infty)$, then

$$\begin{aligned} & 0 < x^{3/2}(f''(x) + f'(x)) < x^{3/2} f''(x) \\ \Rightarrow\ & 0 < \lim_{n\to\infty} n(\ddot{\mathcal{T}}_n(f; x) - f(x)) < x \lim_{n\to\infty} n(\mathcal{T}(f; x) - f(x)) \\ \Rightarrow\ & f(x) < \ddot{\mathcal{T}}_n(f; x) < \mathcal{T}(f; x). \end{aligned}$$

This is the required result.

Example 2.2.10 *This illustration graphically demonstrates that if a function f satisfies equation (2.46), then the newly constructed operators $\ddot{\mathcal{T}}_n$ have better convergence than the original operators $\mathcal{T}_n$. We can check that for the function $f(x) = e^{-5x}$, (2.46) holds with strict inequalities. In the following Figure, we have drawn the graph of f (Gray), $\ddot{\mathcal{T}}_n$ (Green), $\mathcal{T}_n$ (Orange) and in the following Table we have estimated the error for the operators $\ddot{\mathcal{T}}_n$ and $\mathcal{T}_n$. One can easily see from Figure 2.3 and Table 2.2 that $\ddot{\mathcal{T}}_n$ converges better than $\mathcal{T}_n$ for the class of functions which satisfies (2.46).*

Table 2.2: Evaluation of error for the operators $\ddot{\mathcal{T}}_n$ and $\mathcal{T}_n$

$x \to$	0.5	1	1.5	2
$\left\lvert\ddot{\mathcal{T}}_n(f; x) - f(x)\right\rvert$	0.0819647	0.0607227	0.0322153	0.0154151
$\lvert\mathcal{T}_n(f; x) - f(x)\rvert$	0.125698	0.103568	0.0636199	0.0364577

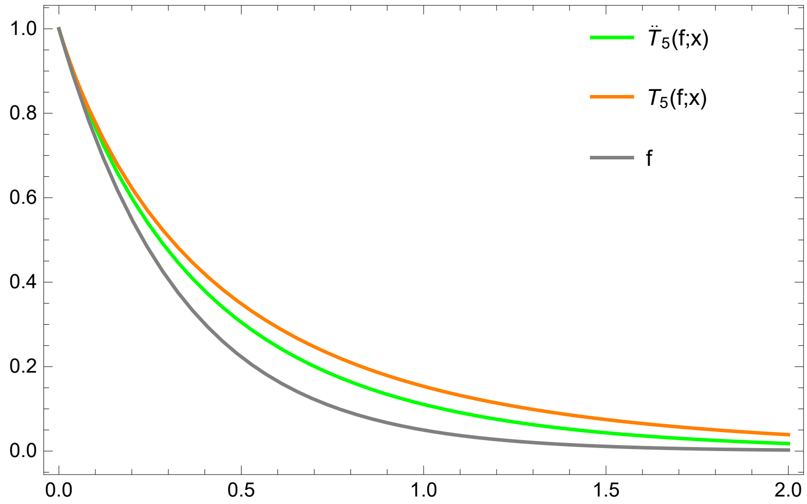


Figure 2.3: Comparison of modified operators $\ddot{\mathcal{T}}_n$ with original operators $\mathcal{T}_n$

2.2.6 Convergence Graphs and Error Estimation Table

Example 2.2.11 *The function* $f(x) = 5\cos(x) - xe^{\frac{x}{20}}$ *is approximated using the operators* $\ddot{\mathcal{T}}_n$ *for* $n = 10, 20, 100$*. The resulting approximations are shown in Fig 2.4. We also estimated the absolute error* $\ddot{E}_n = \left|\ddot{\mathcal{T}}_n(f;x) - f(x)\right|$ *for different values of n in Table 2.3 and presented the corresponding graph in Fig 2.5. From the figures and the Table, it is evident that the proposed operators (2.35) converges to* $f(x)$ *as the value of n increases.*

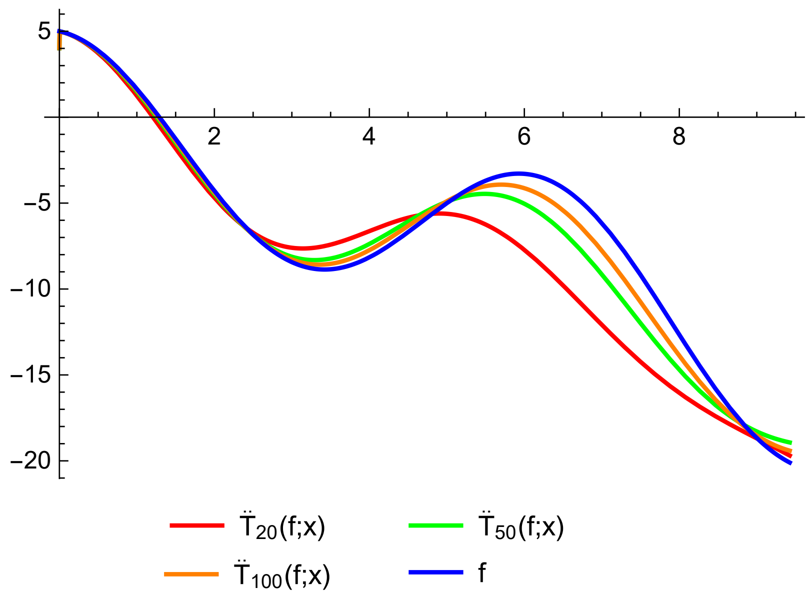


Figure 2.4: The convergence of operators $\ddot{\mathcal{T}}_n$ to the function $f(x) = 5\cos(x) - xe^{\frac{x}{20}}$ for $n = 20, 50, 100$

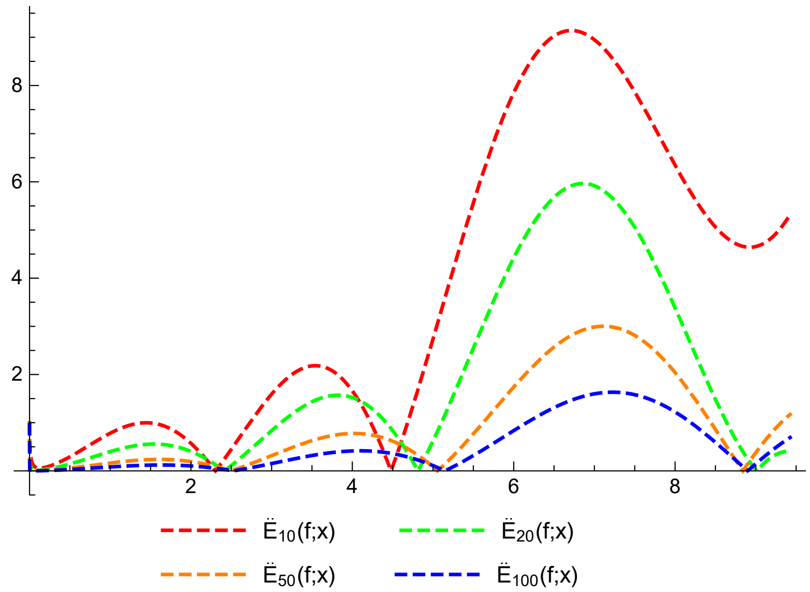


Figure 2.5: Graphical representation of absolute error of operators $\ddot{\mathcal{T}}_n$ to the function $f(x) = 5\cos(x) - xe^{\frac{x}{20}}$ for $n = 10, 20, 50, 100$.

Table 2.3: Evaluation of error for $f(x) = 5\cos(x) - xe^{\frac{x}{20}}$ for $n = 10, 20, 50, 100$.

x	$\ddot{E}_{10}$	$\ddot{E}_{20}$	$\ddot{E}_{50}$	$\ddot{E}_{100}$
$\pi/10$	0.140182	0.0670473	0.0264392	0.0131557
$2\pi/10$	0.416172	0.205997	0.0817337	0.0407404
$3\pi/10$	0.730007	0.371476	0.149512	0.074855
$4\pi/10$	0.954686	0.507173	0.208961	0.10539
$5\pi/10$	0.977187	0.555486	0.237572	0.121236
$6\pi/10$	0.731075	0.472088	0.216107	0.112624
$7\pi/10$	0.219531	0.238647	0.133384	0.0734303
$8\pi/10$	0.479217	0.129237	0.0100886	0.00302205
$9\pi/10$	1.22514	0.58074	0.200355	0.0927286
π	1.84462	1.0357	0.410562	0.201475

Chapter 3

Approximation properties of modified Gamma operators preserving t^{ϑ}

Researchers have spent the last few decades studying a large array of approximation operators due to the development of theory of gamma function. This chapter focuses mainly on the investigation of a modification of certain Gamma-type operators. These operators preserve the test functions $e_{\vartheta} = t^{\vartheta}, \vartheta \in \mathbb{N}$ and it can be observed that the best approximation is attained while preservation of the test function e_3. We have investigated the approximation properties of these operators in the sense of the usual modulus of continuity and Peetre's K-functional. Further, the degree of approximation is also established for the function of bounded variation. The results are validated with some figures and error table.

3.1 Introduction

A remarkable and well-known result in mathematical analysis is the Weierstrass approximation theorem, which states that every continuous function defined on a closed interval $[a, b]$ can be approximated uniformly as closely as possible by a polynomial function. Bohmann-Korovkin theorem is an important approach that guarantees for the positive linear operators to converge to the desired continuous function uniformly on the compact interval $[a, b]$. Gadijev [81] extended the Korovkin theorem on the unbounded interval $[0, \infty)$. Korovkin theorems are one of the most powerful criteria to determine the approximation process of positive linear operators. In continuation of the idea of constructing new approximation processes, King [117] considered an effective technique to modify

Bernstein operators that preserve the test function t^2 and showed that the order of approximation of these King type Bernstein operators is at least as good as the order of approximation by the Bernstein polynomial whenever $0 \leq x < 1/3$. This technique was later used to modify several operators. We mention important papers in this direction due to Acar et al. [5], Acar et al. [6], Acar et al. [10], Acar et al. [11], Agratini-Tarabie [20], Bodur et al. [40], Deo-Bhardwaj [63], Deo-Dhamija [58], Duman-Özarslan [76], Gupta-Aral [87], Gupta-Holhoş [90], Lipi-Deo [122], Mishra-Deo [135], Nur Deveci et al. [138], Ozsarac-Acar [141].

In 1967, Lupas and Müller [127] introduced the sequence of positive linear operators on the interval $(0,\infty)$, called Gamma operators, defined by

$$G_n(f;x) = \int_0^\infty g_n(x,z) f\left(\frac{n}{z}\right) dz, \tag{3.1}$$

where $g_n(x,z) = e^{-zx}\frac{x^{n+1}z^n}{\Gamma(n+1)}$.

A detailed study of approximation of functions by these operators was done by authors of the following articles [55; 127; 130; 132; 173]. Over the next few years, a widespread research had been carried out to refine these operators to obtain new operators. Some of these new operators possessed similar approximation properties (see [105; 128; 133]) while others produced modifications of their classical counterparts (see [12; 120; 168]). Mazhar [130] introduced operators based upon the basis function of operators (3.1) and defined it in the following way:

$$F_n(f;x) = \int_0^\infty g_n(x,z)\, dz \int_0^\infty g_{n-1}(z,t) f(t)\, dt \tag{3.2}$$

for $n > 1,\ x \in (0,\infty)$ and the condition that the integral on the right side converges to some function f. Karsli [115] introduced another form of operators (3.2) that preserves constants and the test functions $f(t) = t^2$. The author estimated the approximation rate for functions having derivatives of bounded variation on the interval $(0,\infty)$. These operators are defined as:

$$\begin{aligned} F_n(f;x) &= \int_0^\infty g_{n+2}(x,z)\, dz \int_0^\infty g_n(z,t) f(t)\, dt \\ &= \frac{(2n+3)!x^{n+3}}{n!\,(n+2)!} \int_0^\infty \frac{t^n}{(x+t)^{2n+4}} f(t)\, dt. \end{aligned} \tag{3.3}$$

Motivated by the above mentioned researches, we aim to construct a modification of operators (3.3), which preserve the test functions $t^\vartheta, \vartheta \in \mathbb{N}$. Let $\beta_n^{(\vartheta)} \in [0,\infty)$, then we consider the operators:

$$\mathbb{G}_n^{(\vartheta)}(f;x) = \frac{(2n+3)!\left(\beta_n^{(\vartheta)}(x)\right)^{n+3}}{n!\,(n+2)!}\int_0^\infty \frac{t^n}{\left(\beta_n^{(\vartheta)}(x)+t\right)^{2n+4}}f(t)dt, \tag{3.4}$$

such that

$$\mathbb{G}_n^{(\vartheta)}(f;x) = F\left(f;\beta_n^{(\vartheta)}(x)\right)$$

and $\beta_n^{(\vartheta)}(x)$ will be defined later on.

If the above operators preserve the function t^ϑ, for fixed $\vartheta \in \mathbb{N}$, then

$$\begin{aligned}\mathbb{G}_n^{(\vartheta)}\left(t^\vartheta;x\right) = x^\vartheta &= \frac{(2n+3)!\left(\beta_n^{(\vartheta)}(x)\right)^{n+3}}{n!\,(n+2)!}\int_0^\infty \frac{t^{n+\vartheta}dt}{\left(\beta_n^{(\vartheta)}(x)+t\right)^{2n+4}}\\ &= \left(\beta_n^{(\vartheta)}(x)\right)^\vartheta \frac{(n+\vartheta)!\,(n+2-\vartheta)!}{n!\,(n+2)!}\\ &= \left(\beta_n^{(\vartheta)}(x)\right)^\vartheta \frac{(n+1)_\vartheta}{(n-\vartheta+3)_\vartheta}\end{aligned}$$

implying

$$\beta_n^{(\vartheta)}(x) = \left(\frac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\right)^{1/\vartheta} x, \tag{3.5}$$

where Pochhammer polynomial (sometime called rising factorial or ascending factorial) is given as $(n)_\vartheta = n(n+1)\dots(n+\vartheta-1)$, $(n)_0 = 1$. Thus modified operators $\mathbb{G}_n^{(\vartheta)}, \vartheta \in \mathbb{N}$ becomes

$$\mathbb{G}_n^{(\vartheta)}(f;x) = \frac{(2n+3)!x^{n+3}}{n!\,(n+2)!}\left(\frac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\right)^{n+3/\vartheta}\int_0^\infty \frac{t^n}{\left(\left(\frac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\right)^{1/\vartheta}x+t\right)^{2n+4}}f(t)\,dt.$$

The modified operators $\mathbb{G}_n^{(\vartheta)}$ preserve constant function along with the function $t^\vartheta, \vartheta \in \mathbb{N}$. In particular, modified operators (3.4) reduce to original operators (3.3) for $\vartheta = 2$.

3.2 Preliminaries

Lemma 3.2.1 *Let the μ-th order moment $\mathbb{G}_n^{(\vartheta)}(t^\mu;x)$ with $\mu \in \{0\}\bigcup\mathbb{N}$ satisfies the following relation*

$$\mathbb{G}_n^{(\vartheta)}(t^\mu;x) = \frac{(n+1)_\mu}{(n-\mu+3)_\mu}\left(\beta_n^{(\vartheta)}(x)\right)^\mu = \frac{(n+1)_\mu}{(n-\mu+3)_\mu}\left[\frac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\right]^{\mu/\vartheta} x^\mu.$$

Some initial moments are

$$\mathbb{G}_n^{(\vartheta)}(1;x) = 1;$$

$$\mathbb{G}_n^{(\vartheta)}(t;x) = \tfrac{(n+1)}{(n+2)}\beta_n^{(\vartheta)}(x) = \tfrac{(n+1)}{(n+2)}\left[\tfrac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\right]^{1/\vartheta} x;$$

$$\mathbb{G}_n^{(\vartheta)}\left(t^2;x\right) = \left(\beta_n^{(\vartheta)}(x)\right)^2 = \left[\tfrac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\right]^{2/\vartheta} x^2;$$

$$\mathbb{G}_n^{(\vartheta)}\left(t^3;x\right) = \left(\beta_n^{(\vartheta)}(x)\right)^3 + \tfrac{3}{n}\left(\beta_n^{(\vartheta)}(x)\right)^3 = \tfrac{(n+1)_3}{(n)_3}\left[\tfrac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\right]^{3/\vartheta} x^3;$$

$$\mathbb{G}_n^{(\vartheta)}\left(t^4;x\right) = \left(\beta_n^{(\vartheta)}(x)\right)^4 + \tfrac{4(2n+3)}{n(n-1)}\left(\beta_n^{(\vartheta)}(x)\right)^4 = \tfrac{(n+1)_4}{(n-1)_4}\left[\tfrac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\right]^{4/\vartheta} x^4.$$

It is clear enough from here that operators $\mathbb{G}_n^{(\vartheta)}$ preserve constant and linear functions when $\vartheta = 1$ and these operators preserve constant and x^2 when $\vartheta = 2$.

Proof: By (3.4) we can write

$$\begin{aligned}
\mathbb{G}_n^{(\vartheta)}(t^\mu;x) &= \frac{(2n+3)!\left(\beta_n^{(\vartheta)}(x)\right)^{n+3}}{n!\,(n+2)!}\int\limits_0^\infty \frac{t^{n+\mu}dt}{\left(\beta_n^{(\vartheta)}(x)+t\right)^{2n+4}} \\
&= \frac{(n+\mu)!\,(n+2-\mu)!}{n!\,(n+2)!}\left(\beta_n^{(\vartheta)}(x)\right)^\mu \\
&= \frac{(n+1)_\mu}{(n-\mu+3)_\mu}\left(\beta_n^{(\vartheta)}(x)\right)^\mu \\
&= \frac{(n+1)_\mu}{(n-\mu+3)_\mu}\left[\left(\frac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\right)^{\mu/\vartheta} x^\mu\right].
\end{aligned}$$

We can get the value of initial moment by substituting $\mu = 0, 1, 2, 3$, and 4.

Lemma 3.2.2 *For the central moment $c_\mu^{\mathbb{G}_n^{(\vartheta)}}(x) = \mathbb{G}_n^{(\vartheta)}((t-x)^\mu;x), \mu \in \{0\}\bigcup\mathbb{N}$ of proposed operators (3.4) the following recursion holds:*

$$\begin{aligned}
(n-\mu+2)\,c_{\mu+1}^{\mathbb{G}_n^{(\vartheta)}}(x) &= \left[n\beta_n^{(\vartheta)}(x) - x(n+4) + (\mu+1)\left(\beta_n^{(\vartheta)}(x)+2x\right)\right]c_\mu^{\mathbb{G}_n^{(\vartheta)}}(x) \\
&\quad + \mu x\left(\beta_n^{(\vartheta)}(x)+x\right)c_{\mu-1}^{\mathbb{G}_n^{(\vartheta)}}(x),
\end{aligned}$$

where $\beta_n^{(\vartheta)}(x)$ is defined in equation (3.5).

Proof: An alternate form of proposed operators (3.4) can be given as

$$\mathbb{G}_n^{(\vartheta)}(f;x) = \int_0^\infty \chi_n^{(\vartheta)}(x,t)f(t)\,dt,$$

where

$$\chi_n^{(\vartheta)}(x,t) = \frac{(2n+3)!\left(\beta_n^{(\vartheta)}(x)\right)^{n+3}}{n!\,(n+2)!}\left(t^n\left(\beta_n^{(\vartheta)}(x)+t\right)^{-2n-4}\right).$$

Differentiating both side of above expression with respect to t, we get

$$\begin{aligned}
t\left(\beta_n^{(\vartheta)}(x)+t\right)\frac{\partial\chi_n^{(\vartheta)}(x,t)}{\partial t} &= \left(n\left(\beta_n^{(\vartheta)}(x)+t\right) + t(-2n-4)\right)\chi_n^{(\vartheta)}(x,t) \\
&= \left(n\beta_n^{(\vartheta)}(x) - x(n+4) - (t-x)(n+4)\right)\chi_n^{(\vartheta)}(x,t).
\end{aligned}$$

Multiplying both sides with $(t-x)^{\mu}$ and then integrating with respect to t from $t=0$ to $t=\infty$, we obtain

$$\int_0^{\infty} t\left(\beta_n^{(\vartheta)}(x)+t\right)\frac{\partial\chi_n^{(\vartheta)}(x,t)}{\partial t}(t-x)^{\mu}dt$$
$$=\int_0^{\infty}\left(n\beta_n^{(\vartheta)}(x)-x(n+4)-(t-x)(n+4)\right)(t-x)^{\mu}\chi_n^{(\vartheta)}(x,t)\,dt$$
$$=\left(n\beta_n^{(\vartheta)}(x)-x(n+4)\right)c_{\mu}^{\mathbb{G}_n^{(\vartheta)}}(x)-(n+4)\,c_{\mu+1}^{\mathbb{G}_n^{(\vartheta)}}(x)\,. \tag{3.6}$$

Making use of

$$t\left(\beta_n^{(\vartheta)}(x)+t\right)=(t-x)^2+\left(\beta_n^{(\vartheta)}(x)+2x\right)(t-x)+x\left(\beta_n^{(\vartheta)}(x)+x\right)$$

in left hand side of (3.6) and integrating partially, we get

$$\int_0^{\infty} t\left(\beta_n^{(\vartheta)}(x)+t\right)\frac{\partial\chi_n^{(\vartheta)}(x,t)}{\partial t}(t-x)^{\mu}dt$$
$$=-(\mu+2)\int_0^{\infty}\chi_n^{(\vartheta)}(x,t)(t-x)^{\mu+1}dt-\left(\beta_n^{(\vartheta)}(x)+2x\right)(\mu+1)\int_0^{\infty}\chi_n^{(\vartheta)}(x,t)(t-x)^{\mu}dt$$
$$-\mu x\left(\beta_n^{(\vartheta)}(x)+x\right)\int_0^{\infty}\chi_n^{(\vartheta)}(x,t)(t-x)^{\mu-1}dt$$
$$=-(\mu+2)\,c_{\mu+1}^{\mathbb{G}_n^{(\vartheta)}}(x)-(\mu+1)\left(\beta_n^{(\vartheta)}(x)+2x\right)c_{\mu}^{\mathbb{G}_n^{(\vartheta)}}(x)-\mu x\left(\beta_n^{(\vartheta)}(x)+x\right)c_{\mu-1}^{\mathbb{G}_n^{(\vartheta)}}(x)\,. \tag{3.7}$$

As a consequence of (3.6) and (3.7), we can write

$$\left(n\beta_n^{(\vartheta)}(x)-x(n+4)\right)c_{\mu}^{\mathbb{G}_n^{(\vartheta)}}(x)-(n+4)\,c_{\mu+1}^{\mathbb{G}_n^{(\vartheta)}}(x)$$
$$=-(\mu+2)\,c_{\mu+1}^{\mathbb{G}_n^{(\vartheta)}}(x)-(\mu+1)\left(\beta_n^{(\vartheta)}(x)+2x\right)c_{\mu}^{\mathbb{G}_n^{(\vartheta)}}(x)-\mu x\left(\beta_n^{(\vartheta)}(x)+x\right)c_{\mu-1}^{\mathbb{G}_n^{(\vartheta)}}(x)\,.$$

Rearranging the terms on both sides, we get the desired relation.
Some initial central moments are given below as a result of above recursion.

$$c_1^{\mathbb{G}_n^{(\vartheta)}}(x)=\left[\frac{(n+1)}{(n+2)}\left(\frac{(n-\vartheta+3)_{\vartheta}}{(n+1)_{\vartheta}}\right)^{1/\vartheta}-1\right]x;$$

$$c_2^{\mathbb{G}_n^{(\vartheta)}}(x)=\left[\left(\frac{(n-\vartheta+3)_{\vartheta}}{(n+1)_{\vartheta}}\right)^{2/\vartheta}-\frac{2(n+1)}{(n+2)}\left(\frac{(n-\vartheta+3)_{\vartheta}}{(n+1)_{\vartheta}}\right)^{1/\vartheta}+1\right]x^2;$$

$$c_3^{\mathbb{G}_n^{(\vartheta)}}(x)=\left[\frac{(n+3)}{n}\left(\frac{(n-\vartheta+3)_{\vartheta}}{(n+1)_{\vartheta}}\right)^{3/\vartheta}-3\left(\frac{(n-\vartheta+3)_{\vartheta}}{(n+1)_{\vartheta}}\right)^{2/\vartheta}\right.$$
$$\left.+\frac{3(n+1)}{(n+2)}\left(\frac{(n-\vartheta+3)_{\vartheta}}{(n+1)_{\vartheta}}\right)^{1/\vartheta}-1\right]x^3;$$

$$c_4^{\mathbb{G}_n^{(\vartheta)}}(x) = \Big[\tfrac{(n+3)(n+4)}{n(n-1)}\Big(\tfrac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\Big)^{4/\vartheta} - \tfrac{4(n+3)}{n}\Big(\tfrac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\Big)^{3/\vartheta} + 6\Big(\tfrac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\Big)^{2/\vartheta} - \tfrac{4(n+1)}{(n+2)}\Big(\tfrac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\Big)^{1/\vartheta} + 1\Big]x^4.$$

$$c_6^{\mathbb{G}_n^{(\vartheta)}}(x) = \Big[\tfrac{(n+3)(n+4)(n+5)(n+6)}{n(n-1)(n-2)(n-3)}\Big(\tfrac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\Big)^{6/\vartheta} - \tfrac{6(n+3)(n+4)(n+5)}{n(n-1)(n-2)}\Big(\tfrac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\Big)^{5/\vartheta} + \tfrac{15(n+3)(n+4)}{n(n-1)}\Big(\tfrac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\Big)^{4/\vartheta} - \tfrac{20(n+3)}{n}\Big(\tfrac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\Big)^{3/\vartheta} + 15\Big(\tfrac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\Big)^{2/\vartheta} - \tfrac{6(n+1)}{(n+2)}\Big(\tfrac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\Big)^{1/\vartheta} + 1\Big]x^6.$$

Generally we have $c_\mu^{\mathbb{G}_n^{(\vartheta)}}(x) = O\Big(\frac{1}{n^{[(\mu+1)/2]}}\Big)$ for $\mu \in \{0\} \bigcup \mathbb{N}$.

3.3 Main Results

In the next result, we estimate the rate of convergence of the proposed operators (3.4) in terms of the modulus of continuity and using this result we prove both numerically and graphically that our modified operators provide best approximation than the corresponding classical one while the test function t^3 is preserved.

Theorem 3.3.1 *For proposed operators $\mathbb{G}_n^{(\vartheta)}$ defined in (3.4) and $f \in C_B[0,\infty)$, we have*

$$\left|\mathbb{G}_n^{(\vartheta)}(f;x) - f(x)\right| \leq C\omega(f;E(\vartheta)),$$

where C is a positive constant and $E(\vartheta)$ is the error function of $\vartheta \in \{0\} \bigcup \mathbb{N}$ defined by $E(\vartheta) = \sqrt{c_2^{\mathbb{G}_n^{(\vartheta)}}(x)}$.

Proof: We begin our proof by mentioning the following property of modulus of continuity:

$$|f(t) - f(x)| \leqslant \left[1 + \frac{|t-x|}{\delta}\right]\omega(f;\delta). \tag{3.8}$$

For all $f \in C_B[0,\infty)$, considering the linearity condition of the operators (3.4) with the fact $\mathbb{G}_n^{(\vartheta)}(1;x) = 1$ and the property (3.8), we get

$$\begin{aligned}\left|\mathbb{G}_n^{(\vartheta)}(f;x) - f(x)\right| &\leq \frac{(2n+3)!\big(\beta_n^{(\vartheta)}(x)\big)^{n+3}}{n!\,(n+2)!}\int_0^\infty \frac{t^n}{\big(\beta_n^{(\vartheta)}(x)+t\big)^{2n+4}}|f(t)-f(x)|\,dt \\ &\leq \omega(f;\delta)\frac{(2n+3)!\big(\beta_n^{(\vartheta)}(x)\big)^{n+3}}{n!\,(n+2)!}\int_0^\infty \frac{t^n}{\big(\beta_n^{(\vartheta)}(x)+t\big)^{2n+4}}\left(1 + \frac{|t-x|}{\delta}\right)dt.\end{aligned}$$

Making use of Cauchy-Schwarz inequality with $\delta = \sqrt{c_2^{\mathbb{G}_n^{(\vartheta)}}(x)}$ and in view of Lemma 3.2.2, we can conclude that

$$\left|\mathbb{G}_n^{(\vartheta)}(f;x) - f(x)\right| \leq \omega(f;\delta)\left(1 + \frac{\sqrt{c_2^{\mathbb{G}_n^{(\vartheta)}}(x)}}{\delta}\right)$$
$$\leq C\omega\left(f;\sqrt{c_2^{\mathbb{G}_n^{(\vartheta)}}(x)}\right).$$

Specifically, for $\vartheta = 1, 2, 3$, we have

$$\left|\mathbb{G}_n^{(1)}(f;x) - f(x)\right| \leq C_1\omega\left(f; x\frac{\sqrt{(2n+3)}}{(n+1)}\right),$$

$$\left|\mathbb{G}_n^{(2)}(f;x) - f(x)\right| \leq C_2\omega\left(f; x\sqrt{\frac{2}{(n+2)}}\right),$$

$$\left|\mathbb{G}_n^{(3)}(f;x) - f(x)\right| \leq C_3\omega\left(f; x\sqrt{\left(\frac{n}{n+3}\right)^{2/3} - \frac{2(n+1)}{(n+2)}\left(\frac{n}{n+3}\right)^{1/3} + 1}\right),$$

where C_1, C_2 and C_3 are constants.
Further, we have

$$\sqrt{\left(\frac{n}{n+3}\right)^{2/3} - \frac{2(n+1)}{(n+2)}\left(\frac{n}{n+3}\right)^{2/3} + 1} \leq \sqrt{\frac{2}{(n+2)}} \leq \frac{\sqrt{(2n+3)}}{(n+1)}.$$

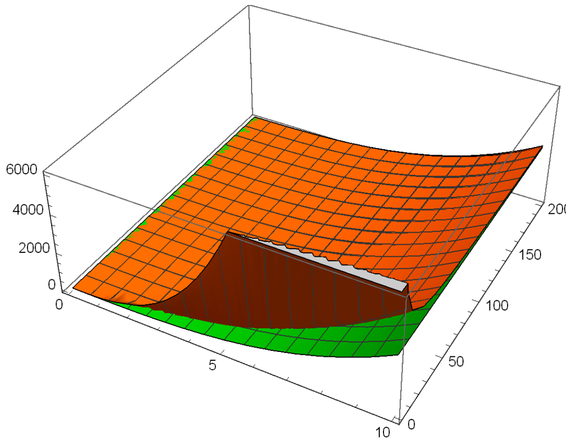
In view of the above theorem, we can observe that the error becomes smaller and decreases monotonically for $x \in (0, \infty)$ and $n \in \mathbb{N}$ until the preservation of test function t^3. For higher order test functions the error starts to increase.

Remark 3.3.2 *As we can see in Table 3.1, the error decreases till $\vartheta = 3$ and gradually begins to increase. Therefore we can draw the conclusion that even though the convergence of the proposed operators takes place in all cases for adequately large n, better approximation is obtained only till the test function t^3 is preserved. For preservation of test functions of higher order we cannot conclude a better approximation.*

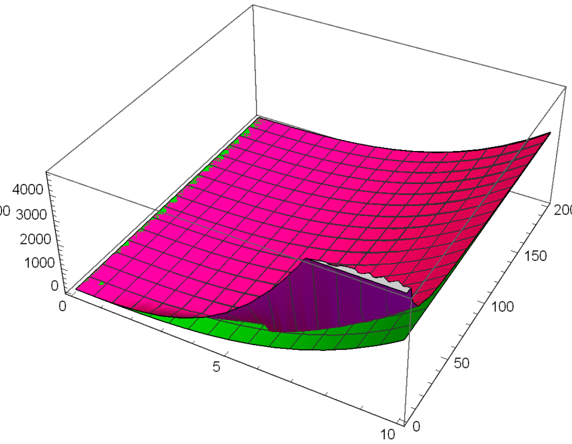
3.3.1 Numerical Analysis

Remark 3.3.3 *From the figures below, we can deduce that operators $\mathbb{G}_n^{(\vartheta)}$ gives better approximation with $\vartheta = 3$ i.e. we get better approximation if these operators preserve t^3.*

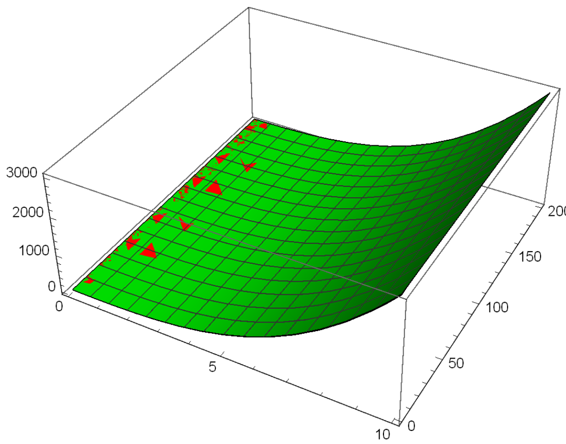
Figure 3.1: Approximation behaviour of $\mathbb{G}_n^{(\vartheta)}$ for the function $f(x) = 3x^3 + \frac{2x^2}{5} + 7$.



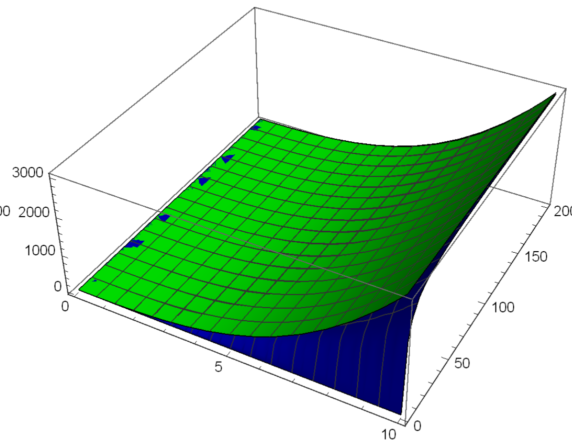
(a) Approximation behaviour of $\mathbb{G}_n^{(1)}$ (Orange) for the function $f(x)$ (Green).



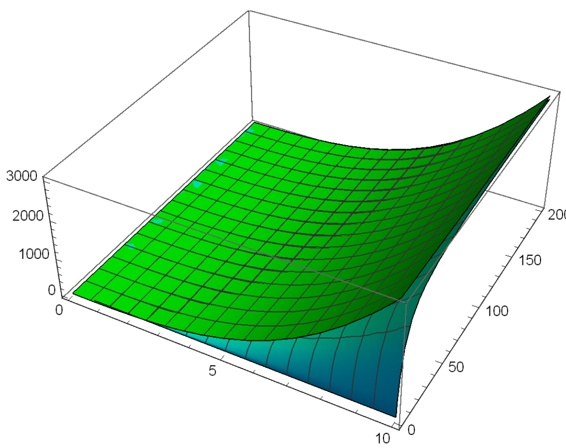
(b) Approximation behaviour of $\mathbb{G}_n^{(2)}$ (Magenta) for the function $f(x)$ (Green).



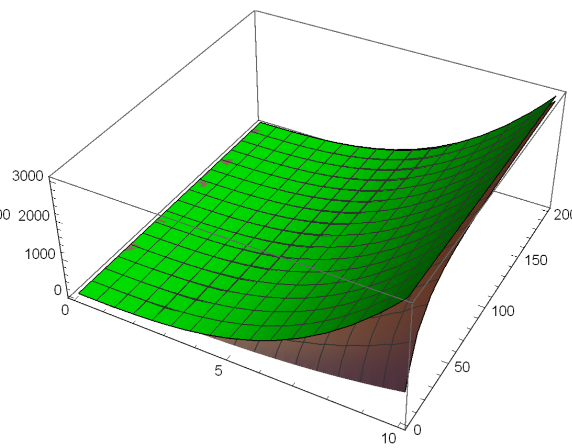
(c) Approximation behaviour of $\mathbb{G}_n^{(3)}$ (Red) for the function $f(x)$ (Green).



(d) Approximation behaviour of $\mathbb{G}_n^{(4)}$ (Blue) for the function $f(x)$ (Green).



(e) Approximation behaviour of $\mathbb{G}_n^{(5)}$ (Cyan) for the function $f(x)$ (Green).



(f) Approximation behaviour of $\mathbb{G}_n^{(6)}$ (Gray) for the function $f(x)$ (Green).

Table 3.1: Error estimation table

n	$E(1)$	$E(2)$	$E(3)$	$E(4)$	$E(5)$	$E(6)$
2	0.881917	0.707107	0.661569	0.704273	1	1
20	0.312259	0.301511	0.298065	0.301247	0.31009	0.32351
200	0.0998749	0.0995037	0.0993805	0.0995025	0.0998653	0.100463
2000	0.0316188	0.031607	0.031603	0.031607	0.0316188	0.0316385
20000	0.00999987	0.0099995	0.00999938	0.0099995	0.00999987	0.0100005
200000	0.00316227	0.00316226	0.00316226	0.00316226	0.00316227	0.00316229
2000000	0.001	0.001	0.000999999	0.000999999	0.001	0.001

In the next result, we examine the degree of approximation in the sense of Peetre's K-functional and weighted approximation.

Theorem 3.3.4 *Let $f \in C_B[0,\infty)$, then there exists a constant $C > 0$ such that*

$$\left|\mathbb{G}_n^{(\vartheta)}(f;x) - f(x)\right| \leq C\omega_2\left(f;\ \sqrt{\delta_n^{(\vartheta)}(x)}\right) + \omega\left(f; \left|\frac{(n+1)}{(n+2)}\beta_n^{(\vartheta)}(x) - 1\right| x\right),$$

where

$$\delta_n^{(\vartheta)}(x) = \left(c_2^{\mathbb{G}_n^{(\vartheta)}}(x) + \left(\frac{(n+1)}{(n+2)}\beta_n^{(\vartheta)}(x) - x\right)^2\right).$$

Proof: We begin by defining an auxiliary sequence of operators

$$\widehat{\mathbb{G}}_n^{(\vartheta)}(f;x) = \mathbb{G}_n^{(\vartheta)}(f;x) - f\left(\frac{(n+1)}{(n+2)}\beta_n^{(\vartheta)}(x)\right) + f(x), \tag{3.9}$$

In view of Lemma 3.2.1, one can observe that above defined operators preserve both constant and linear functions.
Using Taylor's expansion for $x, t \in (0,\infty)$ and $g \in C_B^2[0,\infty)$, where

$$C_B^2[0,\infty) = \{g \in C_B[0,\infty) : g', g'' \in C_B(0,\infty)\}$$

we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-w)g''(w)dw.$$

Applying $\widehat{\mathbb{G}}_n^{(\vartheta)}$ on both sides and using $\widehat{\mathbb{G}}_n^{(\vartheta)}((t-x);x)=0$, we have

$$\begin{aligned}\left|\widehat{\mathbb{G}}_n^{(\vartheta)}(g;x)-g(x)\right| &= \left|\widehat{\mathbb{G}}_n^{(\vartheta)}\left(\int_x^t (t-w)g''(w)dw;x\right)\right| \\ &\leqslant \left|\mathbb{G}_n^{(\vartheta)}\left(\int_x^t (t-w)g''(w)dw;x\right)\right| \\ &\quad + \left|\int_x^{\frac{(n+1)}{(n+2)}\beta_n^{(\vartheta)}(x)} \left(\frac{(n+1)}{(n+2)}\beta_n^{(\vartheta)}(x)-w\right)g''(w)dw\right| \\ &\leqslant \left(c_2^{\mathbb{G}_n^{(\vartheta)}}(x)+\left(\frac{(n+1)}{(n+2)}\beta_n^{(\vartheta)}(x)-x\right)^2\right)\left\|g''\right\| \\ &= \delta_n^{(\vartheta)}(x)\left\|g''\right\|. \end{aligned} \tag{3.10}$$

Also in view of Lemma (3.2.1) we have

$$\left|\mathbb{G}_n^{(\vartheta)}(f;x)\right| \leqslant \frac{(2n+3)!\left(\beta_n^{(\vartheta)}(x)\right)^{n+3}}{n!\,(n+2)!}\int_0^\infty \frac{t^n}{\left(\beta_n^{(\vartheta)}(x)+t\right)^{2n+4}}\left|f(t)\right|dt \leqslant \|f\|.$$

Above inequality along with equation (3.9) implies

$$\left|\widehat{\mathbb{G}}_n^{(\vartheta)}(g;x)\right| \leqslant 3\left\|g\right\|. \tag{3.11}$$

Combining equations (3.9), (3.10) and (3.11), we get

$$\begin{aligned}\left|\mathbb{G}_n^{(\vartheta)}(f;x)-f(x)\right| &\leqslant \left|\widehat{\mathbb{G}}_n^{(\vartheta)}((f-g);x)-(f-g)(x)\right| + \left|\widehat{\mathbb{G}}_n^{(\vartheta)}(g;x)-g(x)\right| \\ &\quad + \left|f(x)-f\left(\frac{(n+1)}{(n+2)}\beta_n^{(\vartheta)}(x)\right)\right| \\ &\leqslant 4\left\|f-g\right\| + \delta_n^{(\vartheta)}(x)\left\|g''\right\| + \omega\left(f;\left|\frac{(n+1)}{(n+2)}\beta_n^{(\vartheta)}(x)-x\right|\right). \end{aligned}$$

Taking infimum over all $g \in C_B^2[0,\infty)$ and using Peetre's K-functional defined by

$$K_2(f;\delta) = \inf_{g\in C_B^2[0,\infty)}\left\{\left\|f-g\right\| + \delta\left\|g''\right\|, \delta>0\right\},$$

we have

$$\begin{aligned}\left|\mathbb{G}_n^{(\vartheta)}(f;x)-f(x)\right| &\leq C\left\{\left\|f-g\right\| + \delta_n^{(\vartheta)}(x)\left\|g''\right\|\right\} + \omega\left(f;\left|\frac{(n+1)}{(n+2)}\beta_n^{(\vartheta)}(x)-x\right|\right) \\ &\leq K_2\left(f;\delta_n^{(\vartheta)}(x)\right) + \omega\left(f;\left|\frac{(n+1)}{(n+2)}\beta_n^{(\vartheta)}(x)-x\right|\right). \end{aligned}$$

Finally using the relation given by Devore and Lorentz in [69], we obtain the required outcome.

3.3.2 Weighted Approximation

We now give the following quantitative Voronovskaja theorem for functions which belongs to weighted space as described in subsection 1.1.5.

Theorem 3.3.5 *If* $f'' \in C_\rho^*[0,\infty)$, *then the following holds true*

$$\left|\mathbb{G}_n^{(\vartheta)}(f;x) - f(x) - \left(\frac{(n+1)}{(n+2)}\left(\frac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\right)^{1/\vartheta} - 1\right)xf'(x)\right.$$
$$\left. -\frac{1}{2}\left(\left(\frac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\right)^{2/\vartheta} - \frac{2(n+1)}{(n+2)}\left(\frac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\right)^{1/\vartheta} - 1\right)x^2 f''(x)\right|$$
$$\leq \frac{8\left(1+x^2\right)}{n}\Omega\left(f''; n^{-1/2}\right).$$

Proof: By Taylor's expansion, we have

$$\mathbb{G}_n^{(\vartheta)}(f;x) - f(x) = \mathbb{G}_n^{(\vartheta)}((t-x);x)f'(x) + \frac{1}{2}\mathbb{G}_n^{(\vartheta)}\left((t-x)^2;x\right)f''(x)$$
$$+ \mathbb{G}_n^{(\vartheta)}\left(g(t,x)(t-x)^2;x\right),$$

where g is a continuous function which vanishes at 0 as t tends to x and is given as $g(t,x) = (f''(y) - f''(x))/2$ with $x < y < t$. Using Lemma 3.2.2, we get

$$\left|\mathbb{G}_n^{(\vartheta)}(f;x) - f(x) - \left(\frac{(n+1)}{(n+2)}\left(\frac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\right)^{1/\vartheta} - 1\right)xf'(x)\right.$$
$$\left. -\frac{1}{2}\left(\left(\frac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\right)^{2/\vartheta} - \frac{2(n+1)}{(n+2)}\left(\frac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\right)^{1/\vartheta} + 1\right)x^2 f''(x)\right|$$
$$\leq \mathbb{G}_n^{(\vartheta)}\left(g(t,x)(t-x)^2;x\right).$$

Making use of inequality $|y-x| \leq |t-x|$, we can have

$$\left|g(t,x)\right| \leq 8\left(1+x^2\right)\left(1+\frac{(t-x)^4}{\delta^4}\right)\Omega(f'';\delta).$$

Then we conclude by Lemma 3.2.2 that

$$\begin{aligned}
&\mathbb{G}_n^{(\vartheta)}\left(g(t,x)(t-x)^2;x\right)\\
&=8\left(1+x^2\right)\Omega(f'';\delta)\left\{c_2^{\mathbb{G}_n^{(\vartheta)}}(x)+\frac{1}{\delta^4}c_6^{\mathbb{G}_n^{(\vartheta)}}(x)\right\}\\
&=8\left(1+x^2\right)\Omega(f'';\delta)\left\{\left(\left(\frac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\right)^{2/\vartheta}-\frac{2(n+1)}{(n+2)}\left(\frac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\right)^{1/\vartheta}-1\right)x^2\right.\\
&+\frac{1}{\delta^4}\left[\frac{(n+3)(n+4)(n+5)(n+6)}{n(n-1)(n-2)(n-3)}\left(\frac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\right)^{6/\vartheta}\right.\\
&-\frac{6(n+3)(n+4)(n+5)}{n(n-1)(n-2)}\left(\frac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\right)^{5/\vartheta}+\frac{15(n+3)(n+4)}{n(n-1)}\left(\frac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\right)^{4/\vartheta}\\
&\left.\left.-\frac{20(n+3)}{n}\left(\frac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\right)^{3/\vartheta}+15\left(\frac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\right)^{2/\vartheta}-\frac{6(n+1)}{(n+2)}\left(\frac{(n-\vartheta+3)_\vartheta}{(n+1)_\vartheta}\right)^{1/\vartheta}+1\right]x^6\right\}.
\end{aligned}$$

Hence the result follows by substituting $\delta=n^{-1/2}$.

3.3.3 Functions of Bounded Variation

Next, we provide an estimate of the rate of pointwise convergence of the proposed operators (3.4) on a Lebesgue point of f which are of bounded variation on $(0,\infty)$ and study the behaviour of operators (3.4) for functions of bounded variation in the interval $(0,\infty)$. For the proof of this theorem, we begin by considering the equivalent form of the operators (3.4) given as:

$$\mathbb{G}_n^{(\vartheta)}(f;x)=\int_0^\infty \chi_n^{(\vartheta)}(x,t)f(t)dt, \tag{3.12}$$

where

$$\chi_n^{(\vartheta)}(x,t)=\frac{(2n+3)!\left(\beta_n^{(\vartheta)}(x)\right)^{n+3}}{n!\,(n+2)!}\frac{t^n}{\left(\beta_n^{(\vartheta)}(x)+t\right)^{2n+4}}.$$

Lemma 3.3.6 *For all $x\in[0,\infty)$, if $0\le y<x$, we have*

$$\Upsilon_n^{(\vartheta)}(x,y)=\int_0^y \chi_n^{(\vartheta)}(x,t)dt\le\frac{2x^2}{n(x-y)^2},$$

and if $x<z<\infty$,

$$1-\Upsilon_n^{(\vartheta)}(x,z)=\int_z^\infty \chi_n^{(\vartheta)}(x,t)dt\le\frac{2x^2}{n(z-x)^2}$$

for sufficiently large n.

Definition 3.3.7 *[47] A Lebesgue point of the function f is a point $x \in \mathbb{R}$ that holds the following condition:*

$$\lim_{j \to 0^+} \frac{1}{j} \int_0^j |f(x+u) - f(x)| du = 0.$$

Theorem 3.3.8 *Let f be a function of bounded variation on every finite subinterval of* $(0, \infty)$ *which satisfies the growth condition given as*

$$|f(t)| \leq Rt^{\alpha},$$

for some constant R and $\alpha > 0$. Then for $s \in \mathbb{N}$ $(2s \geq \alpha)$, $x \in (0, \infty)$, $\epsilon > 0$ and sufficiently large n, we have

$$\left|\mathbb{G}_n^{(\vartheta)}(f;x) - f(x)\right| \leq \frac{4}{n} \sum_{i=1}^{n} \bigvee_{x-\frac{x}{\sqrt{i}}}^{x+\frac{x}{\sqrt{i}}} (f) + \varepsilon \int_{x-\delta}^{x+\delta} \chi_n^{(\vartheta)}(x,t)\, dt + R2^{2s} A(s) \frac{x^{2s}}{n^s}, \; x \in (0, \infty)$$

where $\delta := \frac{x}{\sqrt{n}}$ and $\bigvee_a^b (f)$ represents the total variation of f on $[a, b]$.

Proof: We can write

$$\begin{aligned}
\left|\mathbb{G}_n^{(\vartheta)}(f;x) - f(x)\right| &= \left|\left(\int_0^{x-\frac{x}{\sqrt{n}}} + \int_{x-\frac{x}{\sqrt{n}}}^{x} + \int_x^{x+\frac{x}{\sqrt{n}}} + \int_{x+\frac{x}{\sqrt{n}}}^{\infty}\right) \chi_n^{(\vartheta)}(x,t)(f(t) - f(x))\, dt\right| \\
&\leq \left|\int_0^{x-\frac{x}{\sqrt{n}}} \chi_n^{(\vartheta)}(x,t)(f(t) - f(x))\, dt\right| + \left|\int_{x-\frac{x}{\sqrt{n}}}^{x} \chi_n^{(\vartheta)}(x,t)(f(t) - f(x))\, dt\right| \\
&\quad + \left|\int_x^{x+\frac{x}{\sqrt{n}}} \chi_n^{(\vartheta)}(x,t)(f(t) - f(x))\, dt\right| + \left|\int_{x+\frac{x}{\sqrt{n}}}^{\infty} \chi_n^{(\vartheta)}(x,t)(f(t) - f(x))\, dt\right| \\
&= \left|I_n^{(\vartheta,1)}(x)\right| + \left|I_n^{(\vartheta,2)}(x)\right| + \left|I_n^{(\vartheta,3)}(x)\right| + \left|I_n^{(\vartheta,4)}(x)\right|. \qquad (3.13)
\end{aligned}$$

We begin by computing the integrals $I_n^{(\vartheta,2)}(x)$ and $I_n^{(\vartheta,3)}(x)$ respectively.
Setting

$$F(t) := \int_t^x |f(y) - f(x)|\, dy,$$

then by recalling the definition (3.3.7) of Lebesgue point, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $0 < x - t \leq \delta$,

$$F(t) \leq \varepsilon (x - t). \qquad (3.14)$$

Let us assume that $\delta := \frac{x}{\sqrt{n}}$. Now integrating $I_n^{(\vartheta,2)}(x)$ by parts and using (3.14) we obtain:

$$\begin{aligned}
\left|I_n^{(\vartheta,2)}(x)\right| &= \left|-F(x-\delta)\chi_n^{(\vartheta)}(x, x-\delta) + \int_{x-\delta}^{x} F(t) \frac{\partial}{\partial t}\chi_n^{(\vartheta)}(x,t)\, dt\right| \\
&\leq |-F(x-\delta)|\chi_n^{(\vartheta)}(x, x-\delta) + \int_{x-\delta}^{x} |F(t)| \frac{\partial}{\partial t}\chi_n^{(\vartheta)}(x,t)\, dt \\
&\leq \varepsilon\delta\chi_n^{(\vartheta)}(x, x-\delta) + \varepsilon \int_{x-\delta}^{x} (x-t) \frac{\partial}{\partial t}\chi_n^{(\vartheta)}(x,t)\, dt.
\end{aligned}$$

Again using integration by parts

$$\begin{aligned}\left|I_n^{(\vartheta,2)}(x)\right| &\le \varepsilon\delta\chi_n^{(\vartheta)}(x,x-\delta)+\varepsilon\left\{-\delta\chi_n^{(\vartheta)}(x,x-\delta)+\int_{x-\delta}^{x}\chi_n^{(\vartheta)}(x,t)\,dt\right\}\\ &=\varepsilon\int_{x-\delta}^{x}\chi_n^{(\vartheta)}(x,t)\,dt. \qquad (3.15)\end{aligned}$$

Using analogous approach for $I_n^{(\vartheta,3)}(x)$ we obtain the inequality

$$\left|I_n^{(\vartheta,3)}(x)\right| \le \varepsilon\int_{x}^{x+\delta}\chi_n^{(\vartheta)}(x,t)\,dt. \qquad (3.16)$$

Next we shall use Lebesgue–Stieltjes integral to estimate the integrals $I_n^{(\vartheta,1)}(x)$ and $I_n^{(\vartheta,4)}(x)$ respectively. We begin by estimating $I_n^{(\vartheta,1)}(x)$ using the following Lebesgue–Stieltjes representation:

$$\begin{aligned}I_n^{(\vartheta,1)}(x) &= \int_0^{x-\frac{x}{\sqrt{n}}}(f(t)-f(x))\,d_t\left(\Upsilon_n^{(\vartheta)}(x,t)\right)\\ &=\left(f\left(x-\frac{x}{\sqrt{n}}\right)-f(x)\right)\Upsilon_n^{(\vartheta)}\left(x,x-\frac{x}{\sqrt{n}}\right)-(f(0)-f(x))\,\Upsilon_n^{(\vartheta)}(x,0)\\ &\quad-\int_0^{x-\frac{x}{\sqrt{n}}}\Upsilon_n^{(\vartheta)}(x,t)\,d_t\,(f(t)-f(x)).\end{aligned}$$

Since $\left(f\left(x-\frac{x}{\sqrt{n}}\right)-f(x)\right)\le \bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(f)$, it follows that

$$\left|I_n^{(\vartheta,1)}(x)\right| \le \bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(f)\left|\Upsilon_n^{(\vartheta)}\left(x,x-\frac{x}{\sqrt{n}}\right)\right|+\int_0^{x-\frac{x}{\sqrt{n}}}\left|\Upsilon_n^{(\vartheta)}(x,t)\right|d_t\left(-\bigvee_t^{x}(f)\right).$$

From Lemma (3.3.6), we see that

$$\Upsilon_n^{(\vartheta)}\left(x,x-\frac{x}{\sqrt{n}}\right)\le\frac{2x^2}{n\left(\frac{x}{\sqrt{n}}\right)^2}.$$

Accordingly,

$$\left|I_n^{(\vartheta,1)}(x)\right| \le \frac{2x^2}{n}\frac{1}{\left(\frac{x}{\sqrt{n}}\right)^2}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(f)+\frac{2x^2}{n}\int_0^{x-\frac{x}{\sqrt{n}}}\frac{1}{(x-t)^2}d_t\left(-\bigvee_t^{x}(f)\right).$$

Using integration by parts in the last integral, we have

$$\begin{aligned}\int_0^{x-\frac{x}{\sqrt{n}}}\frac{1}{(x-t)^2}d_t\left(-\bigvee_t^{x}(f)\right) &= -\frac{1}{(x-t)^2}\bigvee_t^{x}(f)\Bigg|_0^{x-\frac{x}{\sqrt{n}}}+\int_0^{x-\frac{x}{\sqrt{n}}}\frac{2}{(x-t)^3}\bigvee_t^{x}(f)\,dt\\ &=-\frac{n}{x^2}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(f)+\frac{1}{x^2}\bigvee_0^{x}(f)+\int_0^{x-\frac{x}{\sqrt{n}}}\frac{2}{(x-t)^3}\bigvee_t^{x}(f)\,dt.\end{aligned}$$

Substituting $t=x-\frac{x}{\sqrt{\lambda}}$ in the last integral, we have

$$\int_0^{x-\frac{x}{\sqrt{n}}}\frac{2}{(x-t)^3}\bigvee_t^{x}(f)\,dt=\frac{1}{x^2}\int_1^{n}\bigvee_{x-\frac{x}{\sqrt{\lambda}}}^{x}(f)\,d\lambda=\frac{1}{x^2}\sum_{i=1}^{n}\bigvee_{x-\frac{x}{\sqrt{i}}}^{x}(f).$$

As a result, we have

$$\left|I_n^{(\vartheta,1)}(x)\right| \le \frac{2x^2}{n}\frac{1}{\left(\frac{x}{\sqrt{n}}\right)^2}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(f)+\frac{2x^2}{n}\left(-\frac{n}{x^2}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(f)+\frac{1}{x^2}\bigvee_{0}^{x}(f)+\frac{1}{x^2}\sum_{i=1}^{n}\bigvee_{x-\frac{x}{\sqrt{i}}}^{x}(f)\right)$$

$$=\frac{2}{n}\left(\bigvee_{0}^{x}(f)+\sum_{i=1}^{n}\bigvee_{x-\frac{x}{\sqrt{i}}}^{x}(f)\right). \tag{3.17}$$

Finally to estimate $I_n^{(\vartheta,4)}(x)$, we introduce the following function:

$$\psi_x(t)=\begin{cases} f(t) & 0\le t\le 2x \\ f(2x) & 2x<t<\infty \end{cases}$$

We rewrite $I_n^{(\vartheta,4)}(x)$ as:

$$\left|I_n^{(\vartheta,4)}(x)\right| = \int_{x+\frac{x}{\sqrt{n}}}^{\infty}\psi_x(t)\,d_t\left(\Upsilon_n^{(\vartheta)}(x,t)\right)+\int_{2x}^{\infty}(f(t)-f(2x))d_t\left(\Upsilon_n^{(\vartheta)}(x,t)\right)$$

$$=: I_n^{(\vartheta,4')}(x)+I_n^{(\vartheta,4'')}(x).$$

Next we evaluate the integral $I_n^{(\vartheta,4')}(x)$ as follows:

$$I_n^{(\vartheta,4')}(x)=\lim_{a\to\infty}\left\{f\left(x+\frac{x}{\sqrt{n}}\right)\left(1-\Upsilon_n^{(\vartheta)}\left(x,x+\frac{x}{\sqrt{n}}\right)\right)\right.$$

$$\left.+\psi_x(a)\left(\Upsilon_n^{(\vartheta)}(x,a)-1\right)+\int_{2x}^{a}f(t)\,d_t\left(\Upsilon_n^{(\vartheta)}(x,t)\right)\right\}.$$

According to Lemma 3.3.6, we obtain

$$I_n^{(\vartheta,4')}(x)=\frac{2x^2}{n}\lim_{a\to\infty}\left\{\frac{n}{x^2}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}(f)+\frac{\psi_x(a)}{(a-x)^2}+\int_0^x\frac{1}{(t-x)^2}d_t\left(\bigvee_x^t(\psi_x)\right)\right\}$$

$$=\frac{2x^2}{n}\left\{\frac{n}{x^2}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}(f)+\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{1}{(t-x)^2}d_t\left(\bigvee_x^t(f)\right)\right\}. \tag{3.18}$$

Integrating by parts the last integral, we have

$$\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{1}{(t-x)^2}d_t\left(\bigvee_x^t(f)\right)=\frac{1}{(t-x)^2}\bigvee_x^t(f)\Big|_{x+\frac{x}{\sqrt{n}}}^{2x}+\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{2}{(x-t)^3}\bigvee_x^t(f)\,dt$$

$$=\frac{1}{x^2}\bigvee_x^{2x}(f)-\frac{n}{x^2}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}(f)+\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{2}{(x-t)^3}\bigvee_x^t(f)\,dt. \tag{3.19}$$

Substituting $t=x+\frac{x}{\sqrt{\lambda}}$ in (3.19), we get

$$\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{2}{(x-t)^3}\bigvee_x^t(f)\,dt=\frac{1}{x^2}\int_1^n\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}(f)\,d\lambda=\frac{1}{x^2}\sum_{i=1}^{n}\bigvee_{x}^{x+\frac{x}{\sqrt{i}}}(f). \tag{3.20}$$

Finally combining (3.18), (3.19), (3.20), we get

$$I_n^{(\vartheta,4')}(x) \leq \frac{4}{n}\left\{\bigvee_x^{2x}(f) + \sum_{i=1}^{n-1}\bigvee_x^{x+\frac{x}{\sqrt{i}}}(f_x)\right\}$$
$$= \frac{4}{n}\sum_{i=1}^{n}\bigvee_x^{x+\frac{x}{\sqrt{i}}}(f).$$

This is the required estimate of $\left|I_n^{(\vartheta,4')}(x)\right|$. Next we proceed to estimate $I_n^{(\vartheta,4'')}(x)$. It is noteworthy that for every $t > 0$, there exists an integer s $(2s > \alpha)$ such that

$$f(t) = O\left(t^{2s}\right).$$

Also for some $\alpha > 0, R > 0$, f satisfies the growth condition $|f(t)| \leq Rt^{\alpha}$ as $t \longrightarrow \infty$. Therefore whenever $t \geq 2x \;\Rightarrow\; 2(t-x) \geq t$, we get

$$I_n^{(\vartheta,4'')}(x) \leq R2^{2s}A(s)\frac{x^{2s}}{n^s}.$$

Combining $I_n^{(\vartheta,4')}(x)$ and $I_n^{(\vartheta,4'')}(x)$, we get

$$\left|I_n^{(\vartheta,4)}(x))\right| \leq \frac{4}{n}\sum_{i=1}^{n}\bigvee_x^{x+\frac{x}{\sqrt{i}}}(f) + R2^{2s}A(s)\frac{x^{2s}}{n^s}. \tag{3.21}$$

Lastly using equations (3.15), (3.16), (3.17) and (3.21) in equation (3.13), we have

$$\left|\mathbb{G}_n^{(\vartheta)}(f;x) - f(x)\right| = \left|I_n^{(\vartheta,1)}(x)\right| + \left|I_n^{(\vartheta,2)}(x)\right| + \left|I_n^{(\vartheta,3)}(x)\right| + \left|I_n^{(\vartheta,4)}(x)\right|$$
$$\leq \frac{2}{n}\left(\bigvee_0^{x}(f) + \sum_{i=1}^{n}\bigvee_{x-\frac{x}{\sqrt{i}}}^{x}(f)\right) + \varepsilon\int_{x-\delta}^{x+\delta}\chi_n^{(\vartheta)}(x,t)\,dt$$
$$+ \frac{4}{n}\sum_{i=1}^{n}\bigvee_x^{x+\frac{x}{\sqrt{i}}}(f) + R2^{2s}A(s)\frac{x^{2s}}{n^s}$$
$$\leq \frac{4}{n}\sum_{i=1}^{n}\bigvee_{x-\frac{x}{\sqrt{i}}}^{x+\frac{x}{\sqrt{i}}}(f) + \varepsilon\int_{x-\delta}^{x+\delta}\chi_n^{(\vartheta)}(x,t)\,dt + R2^{2s}A(s)\frac{x^{2s}}{n^s},$$

which is the required result and the proof is done.

Chapter 4

On generalization of Bernstein operators

Bernstein polynomials, with their helpful structure and applications in various areas (computer technologies, engineering sciences, physics, etc.), have been the subject of intense research for more than a century. A variety of modifications and generalizations of Bernstein polynomials have also been investigated in the literature. This chapter is concerned with the generalization of Bernstein operators. In first section, we propose a Pólya distribution-based generalization of λ-Bernstein operators. We establish some basic results that are relevant for establishing key theorems. We present a theorem and graphical illustrations in support of the proposed operator's interpolation behaviour. In order to illustrate the convergence of proposed operators as well as the effect of changing the parameter "μ", we provide a variety of results and graphs. Second section of this chapter is based on the generalization of Bernstein operators which was defined by Usta in 2020. We begin this section with a fundamental theorem demonstrating the convergence of our newly constructed operators. Also, we derive a theorem determining the degree of approximation in the sense of Peetre's K-functional. We provide a weighted approximation theorem and use the Voronovskaja and Grüss Voronovskaja type theorems to analyse the asymptotic behaviour of our newly constructed operators. We end this chapter by providing a graph and table to validate the convergence and demonstrate the approximation error.

4.1 λ-Bernstein operators based on Pólya distribution

4.1.1 Introduction

The original Pólya-Eggenberger urn model, often known as the Pólya urn, was created in 1923 by Eggenberger and Pólya [78] to explore phenomena like the transmission of infectious diseases. The Pólya-Eggenberger urn model consists of M white balls and N black balls in one of its most basic forms. A ball is picked at random and then replaced with O other balls of the same colour. This process is carried out n times, then the probability of drawing s $(s = 1, 2, ..., n)$ white ball is:

$$\Pr[X = s] = \begin{pmatrix} n \\ s \end{pmatrix} \frac{M(M+O)...[M+(s-1)O]\,N(N+O)...[N+(n-s-1)O]}{(M+N)(M+N+O)...[M+N+(n-1)O]}. \tag{4.1}$$

The distribution described above is referred to as the Pólya-Eggenberger distribution with parameters $(n; M; N; O)$ and includes hypergeometric and binomial distribution as special cases.

Stancu [159] constructed a sequence of positive linear operators using the Pólya-Eggenberger distribution as:

$$S_n^{\langle\mu\rangle}(f;x) = \sum_{k=0}^{n} p_{n,k}^{\langle\mu\rangle}(x)\, f\left(\frac{k}{n}\right), \tag{4.2}$$

where

$$p_{n,k}^{\langle\mu\rangle}(x) = \begin{pmatrix} n \\ k \end{pmatrix} \frac{\prod_{i=0}^{k-1}(x+i\mu)\prod_{i=0}^{n-k-1}(1-x+i\mu)}{\prod_{i=0}^{n-1}(1+i\mu)}$$

and μ is a non-negative parameter that may only be dependent on the natural number n. When $\mu = 0$, operators (4.2) reduce into the classical Bernstein operators [39].

The distribution of the number P of drawings required to obtain n white balls from an urn containing M white balls and N black balls is known as the inverse Pólya-Eggenberger distribution, and it is defined as:

$$\Pr(P = n+s) = \begin{pmatrix} n+s-1 \\ s \end{pmatrix} \frac{M(M+O)...[M+(n-1)O]\,N(N+O)...[N+(s-1)O]}{(M+N)(M+N+O)...[M+N+(n+s-1)O]}, \tag{4.3}$$

for $s \in \mathbb{N} \cup \{0\}$. We direct the readers to [107] in order to provide additional information regarding distributions (4.1) and (4.3).

For a real valued bounded function on $[0, \infty)$ with $0 \leqslant \mu = \mu(n) \to 0$ as $n \to \infty$, Stancu [160] provided the generalization of the Baskakov operators using the inverse Pólya-Eggenberger distribution. For the case $\mu = 0$, these operators reduce into the classical Baskakov operators [35].

Razi [147] developed Bernstein Kantorovich operators based on the Pólya- Eggenberger distribution in 1989 and investigated the rate of convergence and degree of approximation for these operators. Ibrahim [48] introduced Chlodowsky type generalization of Stancu polynomials (also known as Stancu Chlodowsky polynomials) and presented theorems on weighted approximation of functions on the interval $[0, \infty)$. Agrawal et al. [23] introduced the Pólya and Bernstein basis function-based Bézier variant of summation integral type operators. Deo et al. [59] introduced inverse Pólya based Baskakov Kantorovich operators along with its asymptotic formula. The reader is directed to [29; 53; 68; 71; 72; 73] for additional research in this area.

Depending on the parameter λ, Cai et al. [50] proposed and took into consideration a new generalization of Bernstein polynomials known as λ-Bernstein operators. When $\lambda = 0$, these λ-Bernstein operators reduce into the well-known Bernstein operators [39]. Acu et al. [17] defined a Kantorovich form of λ-Bernstein operators and demonstrated how this generalization enhances convergence rate over the classical Kantorovich operators. In order to approximate a function on $[0, 1]$ as well as on its subinterval, Rahman et al. [145] introduced the Kantorovich form of λ-Bernstein operators with shifted knots and demonstrated that these operators approximate the function more accurately than classical Bernstein Kantorovich operators and λ-Bernstein Kantorovich operators. Cai [49] provided the Bézier form of λ-Bernstein Kantorovich operators and derived asymptotic estimate for absolutely continuous function by combining the Bojanic-Cheng decomposition method with a few analysis techniques. Cai and Zhou [51] considered the GBS of the bivariate tensor product of λ -Bernstein Kantorovich operators and established approximation properties of these operators for both B-continuous and B-differentiable functions. Acu et al. [14] considered and investigated a generalization of U_n^ρ operators based on λ-Bernstein operators. The reader is instructed to read [45; 109; 110; 112] for further information on this topic.

In this paper, the generalization of λ-Bernstein operators [50] based on Pólya distribution is presented in the following manner:

$$\mathcal{P}_n^{\langle\lambda,\mu\rangle}(f;x) = \sum_{k=0}^{n} \hat{p}_{n,k}^{\langle\lambda,\mu\rangle}(x)\, f\left(\frac{k}{n}\right), \tag{4.4}$$

where $f \in C[0,1]$, $\lambda \in [-1,1]$, $\mu = \mu(n) \to 0$ as $n \to \infty$ and $\hat{p}_{n,k}^{\langle\lambda,\mu\rangle}(x), k = 0,1,...,n$ are defined below:

$$\begin{cases} \hat{p}_{n,0}^{\langle\lambda,\mu\rangle}(x) = p_{n,0}^{\langle\mu\rangle}(x) - \frac{\lambda}{n+1} p_{n+1,1}^{\langle\mu\rangle}(x), \\ \hat{p}_{n,k}^{\langle\lambda,\mu\rangle}(x) = p_{n,k}^{\langle\mu\rangle}(x) + \lambda \left(\frac{n-2k+1}{n^2-1} p_{n+1,k}^{\langle\mu\rangle}(x) - \frac{n-2k-1}{n^2-1} p_{n+1,k+1}^{\langle\mu\rangle}(x) \right), 1 \leqslant k \leqslant n-1, \\ \hat{p}_{n,n}^{\langle\lambda,\mu\rangle}(x) = p_{n,n}^{\langle\mu\rangle}(x) - \frac{\lambda}{n+1} p_{n+1,n}^{\langle\mu\rangle}(x). \end{cases}$$

Special Cases:

1. For $\lambda = 0$ and $\mu = 0$, proposed operators $\mathcal{P}_n^{\langle\lambda,\mu\rangle}$ transform itno well known Bernstein operators [39].

2. For $\lambda = 0$ and $\mu \neq 0$, these operators $\mathcal{P}_n^{\langle\lambda,\mu\rangle}$ reduces to operators (4.2).

3. For $\lambda \neq 0$ and $\mu = 0$, operators $\mathcal{P}_n^{\langle\lambda,\mu\rangle}$ includes λ-Bernstein operators [50].

4.1.2 Preliminaries

Lemma 4.1.1 *The following equalities hold for the proposed operators $\mathcal{P}_n^{\langle\lambda,\mu\rangle}$ described by equation (4.4):*

$$\mathcal{P}_n^{\langle\lambda,\mu\rangle}(1;x) = 1;$$

$$\mathcal{P}_n^{\langle\lambda,\mu\rangle}(t;x) = x + \lambda \left(\frac{1-2x}{n(n-1)} + \frac{\prod_{i=0}^{n}(x+i\mu) - \prod_{i=0}^{n}(1-x+i\mu)}{n(n-1)\prod_{i=0}^{n}(1+i\mu)} \right);$$

$$\mathcal{P}_n^{\langle\lambda,\mu\rangle}\left(t^2;x\right) = \frac{x^2}{\mu+1} + \frac{x(1+\mu n-x)}{(\mu+1)n} + \lambda \left(\frac{2(1-\mu)x-4x^2}{(\mu+1)n(n-1)} - \frac{1}{n^2(n-1)} + \frac{(1+2n)\prod_{i=0}^{n}(x+i\mu) + \prod_{i=0}^{n}(1-x+i\mu)}{n^2(n-1)\prod_{i=0}^{n}(1+i\mu)} \right).$$

Proof: Form (4.4), it is easy to prove $\mathcal{P}_n^{\langle\lambda,\mu\rangle}(1;x)=1$. Next,

$$
\begin{aligned}
&\mathcal{P}_n^{\langle\lambda,\mu\rangle}(t;x)\\
&=\sum_{k=0}^{n}\hat{p}_{n,k}^{\langle\lambda,\mu\rangle}(x)\frac{k}{n}\\
&=\sum_{k=1}^{n-1}\left\{p_{n,k}^{\langle\mu\rangle}(x)+\lambda\left[\frac{n-2k+1}{n^2-1}p_{n+1,k}^{\langle\mu\rangle}(x)-\frac{n-2k-1}{n^2-1}p_{n+1,k+1}^{\langle\mu\rangle}(x)\right]\right\}\frac{k}{n}\\
&\qquad\qquad+p_{n,n}^{\langle\mu\rangle}(x)-\frac{\lambda}{n+1}p_{n+1,n}^{\langle\mu\rangle}(x)\\
&=\sum_{k=0}^{n}p_{n,k}^{\langle\mu\rangle}(x)\frac{k}{n}+\lambda\left[\sum_{k=0}^{n}p_{n+1,k}^{\langle\mu\rangle}(x)\frac{n-2k+1}{n^2-1}\frac{k}{n}-\sum_{k=1}^{n-1}p_{n+1,k+1}^{\langle\mu\rangle}(x)\frac{n-2k-1}{n^2-1}\frac{k}{n}\right]\\
&=\sum_{k=0}^{n}p_{n,k}^{\langle\mu\rangle}(x)\frac{k}{n}+\lambda\left[\frac{1}{n-1}\sum_{k=0}^{n}p_{n+1,k}^{\langle\mu\rangle}(x)\frac{k}{n}-\frac{2}{n^2-1}\sum_{k=0}^{n}p_{n+1,k}^{\langle\mu\rangle}(x)\frac{k^2}{n}\right.\\
&\qquad\qquad\left.-\frac{1}{n+1}\sum_{k=1}^{n-1}p_{n+1,k+1}^{\langle\mu\rangle}(x)\frac{k}{n}+\frac{2}{n^2-1}\sum_{k=1}^{n-1}p_{n+1,k+1}^{\langle\mu\rangle}(x)\frac{k^2}{n}\right]\\
&=\sum_{k=0}^{n}p_{n,k}^{\langle\mu\rangle}(x)\frac{k}{n}+\lambda\left[\frac{1}{n(n+1)}\sum_{k=0}^{n}p_{n+1,k}^{\langle\mu\rangle}(x)k-\frac{2}{n(n^2-1)}\sum_{k=0}^{n}p_{n+1,k}^{\langle\mu\rangle}(x)k(k-1)\right.\\
&\qquad\qquad-\frac{1}{n(n-1)}\left(\sum_{k=1}^{n-1}p_{n+1,k+1}^{\langle\mu\rangle}(x)(k+1)-\sum_{k=1}^{n-1}p_{n+1,k+1}^{\langle\mu\rangle}(x)\right)\\
&\qquad\qquad\left.+\frac{2}{n(n^2-1)}\sum_{k=1}^{n-1}p_{n+1,k+1}^{\langle\mu\rangle}(x)k(k+1)\right].
\end{aligned}
\tag{4.5}
$$

It is easy to derive the following equalities:

$$\sum_{k=0}^{n} p_{n+1,k}^{\langle\mu\rangle}(x)k = (n+1)\left(x - \frac{\prod_{i=0}^{n}(x+\mu i)}{\prod_{i=0}^{n}(1+\mu i)}\right);$$

$$\sum_{k=0}^{n} p_{n+1,k}^{\langle\mu\rangle}(x)k(k-1) = n(n+1)\left(\frac{x(x+\mu)}{1+\mu} - \frac{\prod_{i=0}^{n}(x+\mu i)}{\prod_{i=0}^{n}(1+\mu i)}\right);$$

$$\sum_{k=1}^{n-1} p_{n+1,k+1}^{\langle\mu\rangle}(x) = 1 - \frac{\prod_{i=0}^{n}(1-x+\mu i)}{\prod_{i=0}^{n}(1+\mu i)} - \frac{(n+1)x\prod_{i=0}^{n-1}(1-x+\mu i)}{\prod_{i=0}^{n}(1+\mu i)} - \frac{\prod_{i=0}^{n}(x+\mu i)}{\prod_{i=0}^{n}(1+\mu i)};$$

$$\sum_{k=1}^{n-1} p_{n+1,k+1}^{\langle\mu\rangle}(x)(k+1) = (n+1)\left(x - \frac{x\prod_{i=0}^{n-1}(1-x+\mu i)}{\prod_{i=0}^{n}(1+\mu i)} - \frac{\prod_{i=0}^{n}(x+\mu i)}{\prod_{i=0}^{n}(1+\mu i)}\right);$$

$$\sum_{k=1}^{n-1} p_{n+1,k+1}^{\langle\mu\rangle}(x)(k+1)k = n(n+1)\left(\frac{x(x+\mu)}{1+\mu} - \frac{\prod_{i=0}^{n}(x+\mu i)}{\prod_{i=0}^{n}(1+\mu i)}\right).$$

Using these equalities in equation (4.5), we get the value of $\mathcal{P}_n^{\langle\lambda,\mu\rangle}(t;x)$. We can also determine the value of $\mathcal{P}_n^{\langle\lambda,\mu\rangle}\left(t^2;x\right)$ in a similar manner.

Lemma 4.1.2 *For $x \in [0,1]$, $\lambda \in [-1,1]$, $\mu = \mu(n) \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} n\mu(n) = l \in \mathbb{R}$, we have*

$$\mathcal{P}_n^{\langle\lambda,\mu\rangle}((t-x);x) = \lambda\left(\frac{1-2x}{n(n-1)} + \frac{\prod_{i=0}^{n}(x+i\mu) - \prod_{i=0}^{n}(1-x+i\mu)}{n(n-1)\prod_{i=0}^{n}(1+i\mu)}\right);$$

$$\mathcal{P}_n^{\langle\lambda,\mu\rangle}\left((t-x)^2;x\right) = \frac{(1+\mu n)(1-x)x}{(\mu+1)n} + \lambda\left(\frac{4\mu x(x-1)}{(\mu+1)n(n-1)} - \frac{1}{n^2(n-1)} + \frac{(1+2n(1-x))\prod_{i=0}^{n}(x+i\mu) + (1+2nx)\prod_{i=0}^{n}(1-x+i\mu)}{n^2(n-1)\prod_{i=0}^{n}(1+i\mu)}\right).$$

Furthermore,

$$\lim_{n\to\infty} \mathcal{P}_n^{\langle\lambda,\mu\rangle}((t-x);x) = 0;$$

$$\lim_{n\to\infty} n\mathcal{P}_n^{\langle\lambda,\mu\rangle}\left((t-x)^2;x\right) = (l+1)(1-x)x.$$

Proof: By substituting the values from Lemma 4.1.1, we can easily prove this Lemma. Throughout the paper, let us define $\varphi_{n,1}^{\langle\lambda,\mu\rangle}(x) = \mathcal{P}_n^{\langle\lambda,\mu\rangle}(t;x)$, $\varphi_{n,2}^{\langle\lambda,\mu\rangle}(x) = \mathcal{P}_n^{\langle\lambda,\mu\rangle}\left(t^2;x\right)$, $\delta_{n,1}^{\langle\lambda,\mu\rangle}(x) = \mathcal{P}_n^{\langle\lambda,\mu\rangle}((t-x);x)$ and $\delta_{n,2}^{\langle\lambda,\mu\rangle}(x) = \mathcal{P}_n^{\langle\lambda,\mu\rangle}\left((t-x)^2;x\right)$.

4.1.3 Interpolation Property

Remark 4.1.3 *For $\lambda \in [-1,1]$, $\mu = \mu(n) \to 0$ as $n \to \infty$ and $x \in [0,1]$, the proposed operators $\mathcal{P}_n^{\langle\lambda,\mu\rangle}$ possess the endpoint interpolation property, that is,*

$$\mathcal{P}_n^{\langle\lambda,\mu\rangle}(f;0) = f(0), \mathcal{P}_n^{\langle\lambda,\mu\rangle}(f;1) = f(1).$$

We can establish the proof using the definition of $\mathcal{P}_n^{\langle\lambda,\mu\rangle}$ and the fact that

$$\hat{p}_{n,k}^{\langle\lambda,\mu\rangle}(x) = \begin{cases} 0, & (k\neq 0) \\ 1, & (k=0) \end{cases} \qquad \hat{p}_{n,k}^{\langle\lambda,\mu\rangle}(x) = \begin{cases} 0, & (k\neq n) \\ 1, & (k=n). \end{cases}$$

Example 4.1.4 *Figure 4.1 displays the graphs of $\hat{p}_{3,k}^{\langle\lambda,\mu\rangle}(x)$ for the values of $\lambda = 1, 0$, and -1. Figure 4.2 displays the corresponding $\mathcal{P}_3^{\langle\lambda,\mu\rangle}$ when $f(x) = \left(x - \frac{1}{4}\right)\sin\left(\frac{5\pi x}{2}\right) + \frac{2}{5}$ with $\mu = \mu(n) = \frac{1}{\sqrt{2\pi n}}\left(\frac{e}{n}\right)^n$. The graphs make it evident that $\mathcal{P}_n^{\langle\lambda,\mu\rangle}$ interpolates the end points of the interval $[0,1]$, which is based on the interpolation property of $\hat{p}_{n,k}^{\langle\lambda,\mu\rangle}(x)$.*

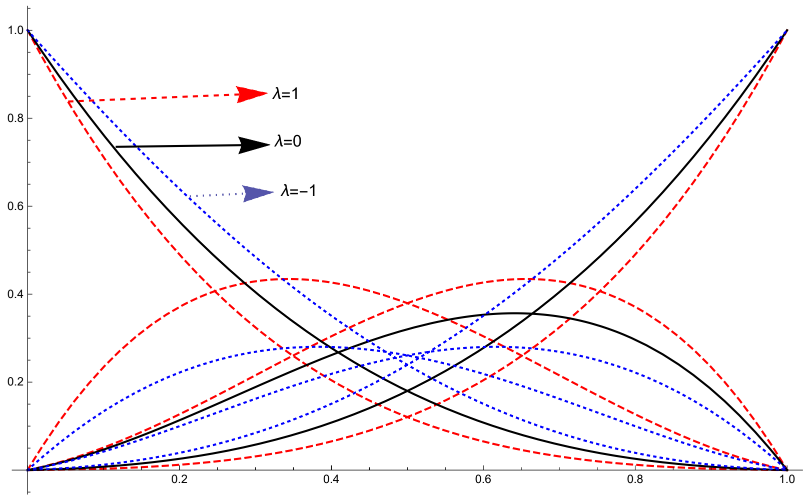


Figure 4.1: The graph of $\hat{p}_{n,k}^{\langle\lambda,\mu\rangle}(x)$ with different value of λ.

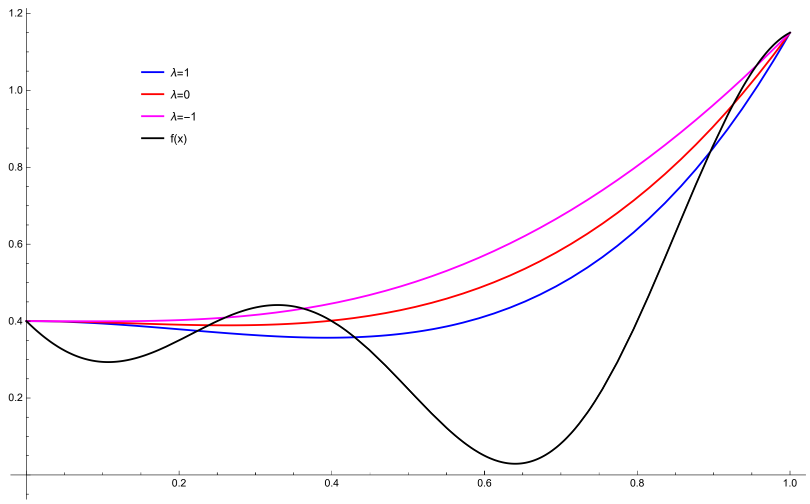


Figure 4.2: Convergence of $\mathcal{P}_3^{\langle -1,\mu\rangle}$ (magenta), $\mathcal{P}_3^{\langle 0,\mu\rangle}$ (red) and $\mathcal{P}_3^{\langle 1,\mu\rangle}$ (blue) with $\mu=\mu(n)=\frac{1}{\sqrt{2\pi n}}\left(\frac{e}{n}\right)^n$ to $f(x)=\left(x-\frac{1}{4}\right)\sin\left(\frac{5\pi x}{2}\right)+\frac{2}{5}$ (black).

4.1.4 Main Results

The smoothness characteristics of the function determine the degree of approximation of positive linear operators, and suitable tools for determining the smoothness of functions are represented by the moduli of continuity of various types. Our subsequent theorems determine the degree of approximation for our proposed operators $\mathcal{P}_n^{\langle\lambda,\mu\rangle}$ in terms of usual and second order modulus of continuity.

Theorem 4.1.5 *Let $\lambda\in[-1,1]$ and $\mu=\mu(n)\to 0$ as $n\to\infty$., then the inequality*

$$\left|\mathcal{P}_n^{\langle\lambda,\mu\rangle}(f;x)-f(x)\right|\leqslant\delta_{n,1}^{\langle\lambda,\mu\rangle}(x)\,|f'(x)|+2\sqrt{\delta_{n,2}^{\langle\lambda,\mu\rangle}(x)}\,\omega\left(f';\sqrt{\delta_{n,2}^{\langle\lambda,\mu\rangle}(x)}\right)$$

holds for $f\in C^1[0,1]$.

Proof: For $f\in C^1[0,1]$ and $x,t\in[0,1]$, we have

$$f(t)-f(x)=(t-x)f'(x)+\int_x^t(f'(y)-f'(x))dy.$$

Applying $\mathcal{P}_n^{\langle\lambda,\mu\rangle}$ on both sides of above mentioned relation, we get

$$\mathcal{P}_n^{\langle\lambda,\mu\rangle}(f(t)-f(x);x)=\mathcal{P}_n^{\langle\lambda,\mu\rangle}((t-x);x)\,f'(x)+\mathcal{P}_n^{\langle\lambda,\mu\rangle}\left(\int_x^t(f'(y)-f'(x))dy;x\right).$$

We know that, $\omega(f;\delta)$ meets the following characteristics:

1. $|f(y)-f(x)|\leqslant\omega(f;|y-x|)$, for any $x\neq y\in[0,1]$,

2. $\omega(f;\lambda\delta) \leqslant (1+\lambda)\,\omega(f;\delta)$, for any $\lambda > 0$.

Using these properties of modulus of continuity, with a few manipulations, we have the relation

$$|f(t) - f(x)| \leqslant \left(1 + \frac{|t-x|}{\delta}\right)\omega(f;\delta), \delta > 0,$$

this implies

$$\left|\int_x^t (f'(y) - f'(x))dy\right| \leqslant \left[|t-x| + \frac{\left|(t-x)^2\right|}{\delta}\right]\omega(f';\delta).$$

Therefore,

$$\left|\mathcal{P}_n^{\langle\lambda,\mu\rangle}(f;x) - f(x)\right| \leqslant \left|\mathcal{P}_n^{\langle\lambda,\mu\rangle}((t-x);x)\right||f'(x)| + \left\{\frac{1}{\delta}\mathcal{P}_n^{\langle\lambda,\mu\rangle}\left((t-x)^2;x\right) + \mathcal{P}_n^{\langle\lambda,\mu\rangle}(|t-x|;x)\right\}\omega(f';\delta).$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left|\mathcal{P}_n^{\langle\lambda,\mu\rangle}(f;x) - f(x)\right| &\leqslant \left|\mathcal{P}_n^{\langle\lambda,\mu\rangle}((t-x);x)\right||f'(x)| \\ &\quad + \sqrt{\mathcal{P}_n^{\langle\lambda,\mu\rangle}\left((t-x)^2;x\right)}\left\{\frac{1}{\delta}\sqrt{\mathcal{P}_n^{\langle\lambda,\mu\rangle}\left((t-x)^2;x\right)} + 1\right\}\omega(f';\delta) \\ &\leqslant \delta_{n,1}^{\langle\lambda,\mu\rangle}(x)|f'(x)| + \sqrt{\delta_{n,2}^{\langle\lambda,\mu\rangle}(x)}\left\{\frac{1}{\delta}\sqrt{\delta_{n,2}^{\langle\lambda,\mu\rangle}(x)} + 1\right\}\omega(f';\delta).\end{aligned}$$

Choosing $\delta = \sqrt{\delta_{n,2}^{\langle\lambda,\mu\rangle}(x)}$, we find the desired inequality.

Theorem 4.1.6 *Let $\lambda \in [-1,1]$ and $\mu = \mu(n) \to 0$ as $n \to \infty$, then the inequality*

$$\left|\mathcal{P}_n^{\langle\lambda,\mu\rangle}(f;x) - f(x)\right| \leqslant 2\omega\left(f;\sqrt{\delta_{n,2}^{\langle\lambda,\mu\rangle}(x)}\right)$$

holds for $f \in C[0,1]$.

Proof: For any $t, x \in [a,b]$, using the following property of modulus of continuity, we get

$$|f(t) - f(x)| \leqslant \left(1 + \frac{(t-x)^2}{\delta^2}\right)\omega(f;\delta).$$

Applying $\mathcal{P}_n^{\langle\lambda,\mu\rangle}$ on both sides of above relation, we get

$$\begin{aligned}\left|\mathcal{P}_n^{\langle\lambda,\mu\rangle}(f;x) - f(x)\right| &\leqslant \mathcal{P}_n^{\langle\lambda,\mu\rangle}(|f(t) - f(x)|;x) \\ &\leqslant \left(1 + \frac{\mathcal{P}_n^{\langle\lambda,\mu\rangle}\left((t-x)^2;x\right)}{\delta^2}\right)\omega(f;\delta).\end{aligned}$$

Choosing $\delta^2 = \delta_{n,2}^{\langle\lambda,\mu\rangle}(x) = \mathcal{P}_n^{\langle\lambda,\mu\rangle}\left((t-x)^2;x\right)$, we obtain the desired result.

Theorem 4.1.7 *For $\lambda \in [-1, 1]$ and $\mu = \mu(n) \to 0$ as $n \to \infty$, then the inequality*

$$\left|\mathcal{P}_n^{\langle\lambda,\mu\rangle}(f;x) - f(x)\right| \leq C\omega_2\left(f; \frac{1}{2}\sqrt{\Upsilon_n^{\langle\lambda,\mu\rangle}(x)}\right) + \omega\left(f; \delta_{n,1}^{\langle\lambda,\mu\rangle}(x)\right)$$

holds for $f \in C[0,1]$, $\Upsilon_n^{\langle\lambda,\mu\rangle}(x) = \delta_{n,2}^{\langle\lambda,\mu\rangle}(x) + \left(\delta_{n,1}^{\langle\lambda,\mu\rangle}(x)\right)^2$ and absolute constant C.

Proof: Consider the operators $\mathbb{P}_n^{\langle\lambda,\mu\rangle}$ defined by

$$\mathbb{P}_n^{\langle\lambda,\mu\rangle}(f;x) = \mathcal{P}_n^{\langle\lambda,\mu\rangle}(f;x) - f\left(\varphi_{n,1}^{\langle\lambda,\mu\rangle}(x)\right) + f(x). \tag{4.6}$$

Due to Lemma 4.1.1 and the fact that these operators are linear in nature, it is obvious that

$$\mathbb{P}_n^{\langle\lambda,\mu\rangle}(1;x) = \mathcal{P}_n^{\langle\lambda,\mu\rangle}(1;x) = 1,$$

$$\mathbb{P}_n^{\langle\lambda,\mu\rangle}(t;x) = \varphi_{n,1}^{\langle\lambda,\mu\rangle}(x) + x - \varphi_{n,1}^{\langle\lambda,\mu\rangle}(x) = x.$$

For $g \in C^2[0,1] = \{f \in C[0,1] : f'' \in C[0,1]\}$, consider the Taylor's formula

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-w)g''(w)dw.$$

Applying $\mathbb{P}_n^{\langle\lambda,\mu\rangle}$ on both sides of above equality and using $\mathbb{P}_n^{\langle\lambda,\mu\rangle}(1;x) = 1$, we get

$$\begin{aligned}\mathbb{P}_n^{\langle\lambda,\mu\rangle}(g;x) &= g(x) + \mathbb{P}_n^{\langle\lambda,\mu\rangle}((t-x);x)\,g'(x) + \mathbb{P}_n^{\langle\lambda,\mu\rangle}\left(\int_x^t (t-w)g''(w)dw; x\right) \\ &= g(x) + \mathcal{P}_n^{\langle\lambda,\mu\rangle}\left(\int_x^t (t-w)g''(w)dw; x\right) - \int_x^{\varphi_{n,1}^{\langle\lambda,\mu\rangle}(x)} \left(\varphi_{n,1}^{\langle\lambda,\mu\rangle}(x) - w\right)g''(w)dw\end{aligned}$$

and hence

$$\begin{aligned}\left|\mathcal{P}_n^{\langle\lambda,\mu\rangle}(g;x) - g(x)\right| &\leqslant \mathcal{P}_n^{\langle\lambda,\mu\rangle}\left(\left|\int_x^t (t-w)g''(w)dw; x\right|\right) + \left|\int_x^{\varphi_{n,1}^{\langle\lambda,\mu\rangle}(x)} \left|\left(\varphi_{n,1}^{\langle\lambda,\mu\rangle}(x) - w\right)\right| |g''(w)|dw\right| \\ &\leqslant \delta_{n,2}^{\langle\lambda,\mu\rangle}(x)\,\|g''\| + \left(\varphi_{n,1}^{\langle\lambda,\mu\rangle}(x) - x\right)^2 \|g''\| \\ &= \left\{\delta_{n,2}^{\langle\lambda,\mu\rangle}(x) + \left(\delta_{n,1}^{\langle\lambda,\mu\rangle}(x)\right)^2\right\}\|g''\| \\ &= \Upsilon_n^{\langle\lambda,\mu\rangle}(x)\,\|g''\|.\end{aligned} \tag{4.7}$$

From relation (4.6), we have

$$\begin{aligned}\left|\mathbb{P}_n^{\langle\lambda,\mu\rangle}(f;x)\right| &\leqslant \left|\mathcal{P}_n^{\langle\lambda,\mu\rangle}(f;x)\right| + |f(x)| + \left|f\left(\varphi_{n,1}^{\langle\lambda,\mu\rangle}(x)\right)\right| \\ &\leqslant \|f\|\,\mathcal{P}_n^{\langle\lambda,\mu\rangle}(1;x) + 2\|f\| = 3\|f\|.\end{aligned} \tag{4.8}$$

Now,

$$\begin{aligned}\left|\mathcal{P}_n^{\langle\lambda,\mu\rangle}(f;x)-f(x)\right| &= \left|\mathbb{P}_n^{\langle\lambda,\mu\rangle}(f;x)-f(x)+f\left(\varphi_{n,1}^{\langle\lambda,\mu\rangle}(x)\right)-f(x)\right| \\ &\leq \left|\mathbb{P}_n^{\langle\lambda,\mu\rangle}(f-g;x)\right|+\left|\mathbb{P}_n^{\langle\lambda,\mu\rangle}(g;x)-g(x)\right|+|f(x)-g(x)|+\left|f\left(\varphi_{n,1}^{\langle\lambda,\mu\rangle}(x)\right)-f(x)\right|.\end{aligned}$$

Using relation (4.7) and (4.8) and definition of modulus of continuity, we have

$$\left|\mathcal{P}_n^{\langle\lambda,\mu\rangle}(f;x)-f(x)\right| \leqslant 4\|f-g\|+\Upsilon_n^{\langle\lambda,\mu\rangle}(x)\|g''\|+\omega\left(f;\varphi_{n,1}^{\langle\lambda,\mu\rangle}(x)-x\right).$$

Applying infimum to all of $g \in C^2[0,1]$, we get

$$\left|\mathcal{P}_n^{\langle\lambda,\mu\rangle}(f;x)-f(x)\right| \leqslant 4K\left(f;\frac{1}{4}\Upsilon_n^{\langle\lambda,\mu\rangle}(x)\right)+\omega\left(f;\delta_{n,1}^{\langle\lambda,\mu\rangle}(x)\right).$$

This concludes the proof in view of relation (1.2).

Our following theorem determines the rate of convergence of the operators $\mathcal{P}_n^{\langle\lambda,\mu\rangle}$ for functions belonging to Lipschitz class $Lip_C(\gamma)$ defined in subsection 1.1.7 for the interval $[0,1]$.

Theorem 4.1.8 *Let $\lambda \in [-1,1], \mu = \mu(n) \to 0$ as $n \to \infty$ and $x \in [0,1]$, then the inequality*

$$\left|\mathcal{P}_n^{\langle\lambda,\mu\rangle}(f;x)-f(x)\right| \leq C\left[\delta_{n,2}^{\langle\lambda,\mu\rangle}\right]^{\frac{\gamma}{2}},$$

holds for $f \in Lip_C(\gamma)$.

Proof: Since $\mathcal{P}_n^{\langle\lambda,\mu\rangle}$ are positive and linear in nature and $f \in Lip_C(\gamma)$, we have

$$\begin{aligned}\left|\mathcal{P}_n^{\langle\lambda,\mu\rangle}(f;x)-f(x)\right| &\leqslant \mathcal{P}_n^{\langle\lambda,\mu\rangle}(|f(t)-f(x)|;x) \\ &= \sum_{k=0}^{n}\hat{p}_{n,k}^{\langle\lambda,\mu\rangle}(x)\left|f\left(\frac{k}{n}\right)-f(x)\right| \\ &\leqslant C\sum_{k=0}^{n}\hat{p}_{n,k}^{\langle\lambda,\mu\rangle}(x)\left|\frac{k}{n}-x\right|^{\gamma} \\ &\leqslant C\sum_{k=0}^{n}\left[\hat{p}_{n,k}^{\langle\lambda,\mu\rangle}(x)\left(\frac{k}{n}-x\right)^2\right]^{\frac{\gamma}{2}}\left[\hat{p}_{n,k}^{\langle\lambda,\mu\rangle}(x)\right]^{\frac{2-\gamma}{2}}.\end{aligned}$$

Applying Hölder's inequality for sums, we obtain

$$\begin{aligned}\left|\mathcal{P}_n^{\langle\lambda,\mu\rangle}(f;x)-f(x)\right| &\leqslant C\left[\sum_{k=0}^{n}\hat{p}_{n,k}^{\langle\lambda,\mu\rangle}(x)\left(\frac{k}{n}-x\right)^2\right]^{\frac{\gamma}{2}}\left[\sum_{k=0}^{n}\hat{p}_{n,k}^{\langle\lambda,\mu\rangle}(x)\right]^{\frac{2-\gamma}{2}} \\ &= C\left[\mathcal{P}_n^{\langle\lambda,\mu\rangle}\left((t-x)^2;x\right)\right]^{\frac{\gamma}{2}}.\end{aligned}$$

This proves theorem 4.1.8.

Finally, we give a Voronovskaja asymptotic formula for $\mathcal{P}_n^{\langle\lambda,\mu\rangle}$.

Theorem 4.1.9 *Let $\lambda \in [-1, 1]$, $\mu = \mu(n) \to 0$ as $n \to \infty$ and $f(x)$ be bounded on $[0, 1]$. Then, for any $x \in (0, 1)$ at which $f''(x)$ exists, we have*

$$\lim_{n\to\infty} n\left[\mathcal{P}_n^{\langle\lambda,\mu\rangle}(f; x) - f(x)\right] = \frac{1}{2}(l+1)(1-x)\,xf''(x).$$

Proof: By the Taylor formula, we may write

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2}(t-x)^2 f''(x) + (t-x)^2 r(t,x), \tag{4.9}$$

where $r(t,x) \in C[0,1]$ is the Peano form of the remainder. Using L'Hopital's rule, we have

$$\lim_{t\to x} r(t,x) = 0.$$

Applying $\mathcal{P}_n^{\langle\lambda,\mu\rangle}$ to (4.9), we obtain

$$\begin{aligned}\lim_{n\to\infty} n\left[\mathcal{P}_n^{\langle\lambda,\mu\rangle}(f; x) - f(x)\right] &= \lim_{n\to\infty} n\mathcal{P}_n^{\langle\lambda,\mu\rangle}((t-x); x)\, f'(x) + \frac{1}{2}\lim_{n\to\infty} n\mathcal{P}_n^{\langle\lambda,\mu\rangle}\left((t-x)^2; x\right) f''(x) \\ &\quad + \lim_{n\to\infty} n\mathcal{P}_n^{\langle\lambda,\mu\rangle}\left((t-x)^2 r(t,x); x\right).\end{aligned} \tag{4.10}$$

By the Cauchy-Schwarz inequality, we have

$$\mathcal{P}_n^{\langle\lambda,\mu\rangle}\left((t-x)^2 r(t,x); x\right) \leqslant \sqrt{\mathcal{P}_n^{\langle\lambda,\mu\rangle}\left((t-x)^4; x\right)}\sqrt{\mathcal{P}_n^{\langle\lambda,\mu\rangle}(r^2(t,x); x)}. \tag{4.11}$$

Since $r^2(x,x) = 0$, then using (4.11), we can obtain

$$\lim_{n\to\infty} n\mathcal{P}_n^{\langle\lambda,\mu\rangle}\left((t-x)^2 r(t,x); x\right) = 0. \tag{4.12}$$

Finally, using (4.10), (4.12) and Lemma 4.1.2, we get

$$\lim_{n\to\infty} n\left[\mathcal{P}_n^{\langle\lambda,\mu\rangle}(f; x) - f(x)\right] = \frac{1}{2}(l+1)(1-x)\,xf''(x).$$

Hence, we get the proof.

4.1.5 Numerical Results

Example 4.1.10 *The convergence of $\mathcal{P}_{15}^{\langle\lambda,\mu\rangle}$ (magenta), $\mathcal{P}_{25}^{\langle\lambda,\mu\rangle}$ (red) and $\mathcal{P}_{45}^{\langle\lambda,\mu\rangle}$ (blue) to $f(x) = \sin(3\sin(3x))$ (black) is illustrated in Figure 4.3 for fixed $\lambda = -0.5$ and $\mu = \mu(n) = \frac{1}{n^5+2^{\log(n)}}$. Table 4.1 computes the absolute error $\mathcal{E}_n^{\langle\lambda,\mu\rangle}(x) = \left|\mathcal{P}_n^{\langle\lambda,\mu\rangle}(f; x) - f(x)\right|$ of the function f for various values of x in the interval [0, 1], and Figure 4.4 displays this error graphically. When n rises from 15 to 45, we notice that the approximation of f by $\mathcal{P}_n^{\langle\lambda,\mu\rangle}$ gets better and error also continues to decrease.*

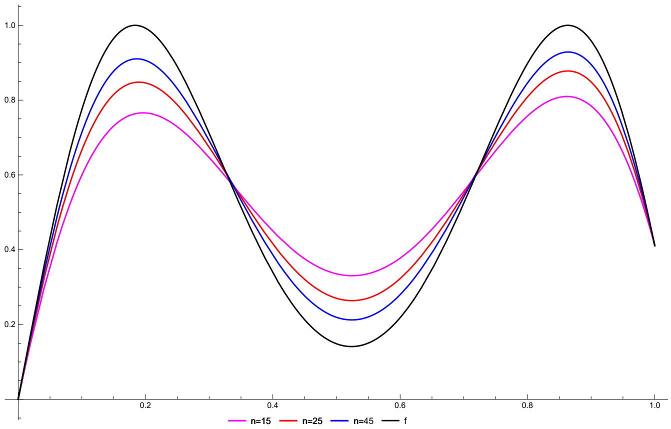


Figure 4.3: Convergence of $\mathcal{P}_{15}^{\langle\lambda,\mu\rangle}$ (magenta), $\mathcal{P}_{25}^{\langle\lambda,\mu\rangle}$ (red) and $\mathcal{P}_{45}^{\langle\lambda,\mu\rangle}$ (blue) for fixed $\lambda = -0.5$ and $\mu = \mu(n) = \frac{1}{n^5+2^{\log(n)}}$ to $f(x) = \sin(3\sin(3x))$ (black).

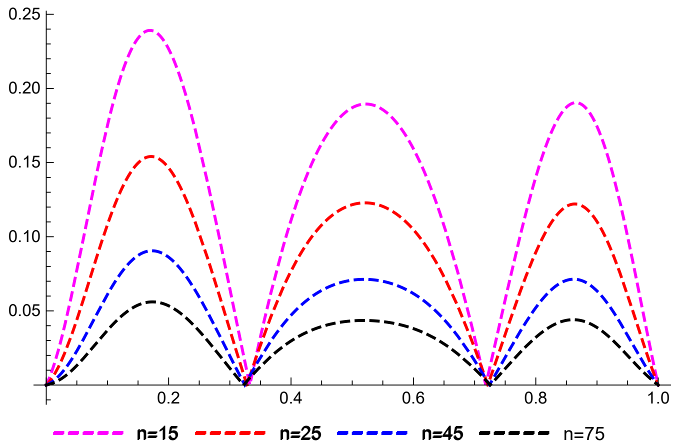


Figure 4.4: Graph of $\mathcal{E}_{15}^{\langle\lambda,\mu\rangle}(x)$ (magenta), $\mathcal{E}_{25}^{\langle\lambda,\mu\rangle}(x)$ (red), $\mathcal{E}_{45}^{\langle\lambda,\mu\rangle}(x)$ (blue) and $\mathcal{E}_{75}^{\langle\lambda,\mu\rangle}(x)$ (black) with $\lambda = -0.5$ and $\mu = \mu(n) = \frac{1}{n^5+2^{\log(n)}}$ for $f(x) = \sin(3\sin(3x))$.

Table 4.1: Estimation of error for various value of x in the interval [0, 1]

x	$\mathcal{E}_{15}^{\langle\lambda,\mu\rangle}$	$\mathcal{E}_{25}^{\langle\lambda,\mu\rangle}$	$\mathcal{E}_{45}^{\langle\lambda,\mu\rangle}$	$\mathcal{E}_{75}^{\langle\lambda,\mu\rangle}$
0.1	0.175061	0.108151	0.0610693	0.0369243
0.2	0.226676	0.146409	0.0861958	0.0534319
0.3	0.0650047	0.0373058	0.0196547	0.0113319
0.4	0.111161	0.0768502	0.0470771	0.0297085
0.5	0.187009	0.121607	0.0707124	0.0432592
0.6	0.159335	0.104776	0.0616742	0.038047
0.7	0.03193	0.0248162	0.0164783	0.0108634
0.8	0.140273	0.0895243	0.0521938	0.0321723
0.9	0.17437	0.110194	0.0634311	0.0387883

Example 4.1.11 *Figure 4.5 shows the graph for the operators* $\mathcal{P}_n^{\langle\lambda,\mu\rangle}$ *for two different sequences* $\mu = \mu(n) = \frac{1}{n!}$ *(red) and* $\mu = \mu(n) = \frac{1}{n\log(n)}$, *(magenta) while keeping* $n = 30$ *and* $\lambda = 0.5$ *fixed for the function* $f(x) = x^4 - \frac{12x^3}{5} + \frac{193x^2}{100} - \frac{57x}{100} + \frac{3}{50}$ *(black). Figure 4.6 shows the graph for the operators* $\mathcal{P}_n^{\langle\lambda,\mu\rangle}$ *for the function* $f(x) = 10x + 2\cos(10x)$ *(black) with fixed* $n = 20$ *and* $\lambda = 0.1$ *for two different sequences* $\mu = \mu(n) = \frac{1}{n^2}$ *(red) and* $\mu = \mu(n) = \frac{1}{n}$, *(magenta). For these two cases, the graphs make it evident that convergence of the operators towards the function occurs best for the sequence with higher rate of convergence.*

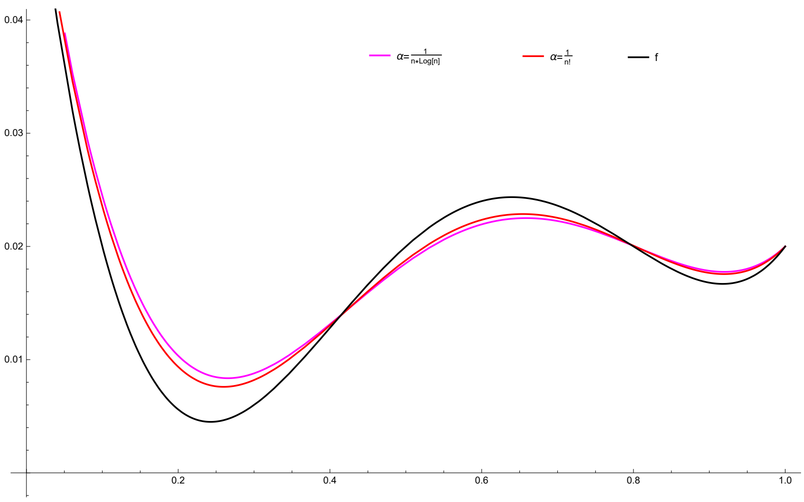


Figure 4.5: Convergence of $\mathcal{P}_{30}^{\left\langle 0.5,\frac{1}{n\log(n)}\right\rangle}$ (magenta) and $\mathcal{P}_{30}^{\left\langle 0.5,\frac{1}{n!}\right\rangle}$ (red) to $f(x) = x^4 - \frac{12x^3}{5} + \frac{193x^2}{100} - \frac{57x}{100} + \frac{3}{50}$ (black).

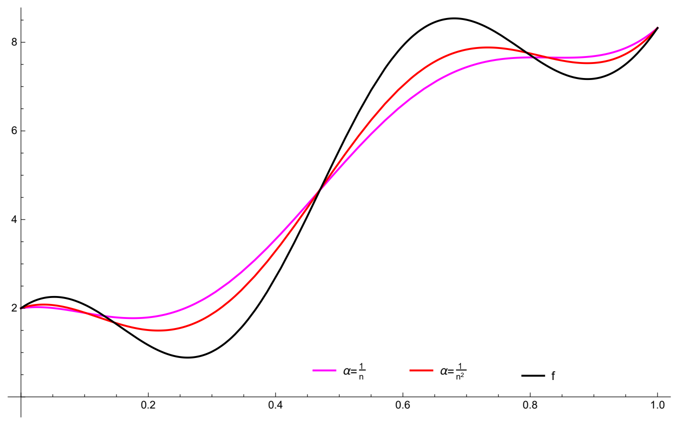


Figure 4.6: Convergence of $\mathcal{P}_{20}^{\left\langle 0.1,\frac{1}{n}\right\rangle}$ (magenta) and $\mathcal{P}_{20}^{\left\langle 0.1,\frac{1}{n^2}\right\rangle}$ (red) to $f(x) = 10x + 2\cos(10x)$ (black).

4.2 Approximation by means of modified Bernstein operators with shifted knot

4.2.1 Introduction

The major goal of approximation theory is to define how to express any function in terms of simpler, more practical functions. Bernstein [39] provided the following definition for the demonstration of the Weierstrass approximation theorem in 1912, which was referred to by his name. For every bounded function on $[0,1]$, $n \geq 1$ and $x \in [0,1]$, Bernstein polynomials are further defined as:

$$\mathcal{B}_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right),$$

A variety of modifications and generalizations of Bernstein polynomials have also been investigated in the literature. One of the main goals of these modifications is to improve approximation speed and reduce the absolute error that occur as a natural outcome of the approximation process. Gadjiev and Ghorbanalizadeh [82] carried out one of these research and defined Bernstein-Stancu type polynomials with shifted knots as follows:

$$G_n^{\langle a_i,b_i\rangle}(f;x) = \left(\frac{n+b_2}{n}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left(x - \frac{a_2}{n+b_2}\right)^k \left(\frac{n+a_2}{n+b_2} - x\right)^{n-k} f\left(\frac{k+a_1}{n+b_1}\right), \tag{4.13}$$

where $\frac{a_2}{n+b_2} \leqslant x \leqslant \frac{n+a_2}{n+b_2}$, and $a_p,\ b_p,\ p = 1,2$ are positive reals satisfying the condition $0 \leqslant a_2 \leqslant a_1 \leqslant b_1 \leqslant b_2$. It is evident that for the case where $a_1 = a_2 = b_1 = b_2 = 0$, we have Bernstein polynomials and if $a_2 = b_2 = 0$, the Bernstein-Stancu polynomials are produced.

Numerous authors have constructed operators with moving intervals of convergence in response to this Gadjiev's research. İçöz [101] introduced Bernstein-Stancu Kantorovich type operators with shifted knot and gave r-th order generalization of these operators. An entirely new class of Bernstein-Durrmeyer operators on movable interval was defined by Acar et al. [7]. They modelled these operators as hypergeometric series and investigated their approximation properties. They used a King type technique to give improved error estimation for these operators. Jiang and Yu [106] explored convergence properties for analytical functions in the movable compact disc and developed two types of complex Kantorovich-Stancu operators and complex Bernstein-Stancu operators. Rahman et al. [145] introduced λ-Bernstein Kantorovich operators with shifted knots and provided graphics and error estimation tables for comparison. A bivariate generalization of the operators discussed in [145] was defined by Agrawal et al. [21]. Q-analogue of Lupaş Bernstein operators [136] and Bernstein operators [137] with shifted knots were also introduced. Rahman [146] established a Kantorovich variant of Lupaş operators based on the Pólya distribution with shifted knots, as well as a bivariate generalization of these operators. Bawa [37] continued the research of these operators and investigated statistical convergence.

Usta [167] proposed a new modification of Bernstein operators which fix constant and preserve Korovkin's other test functions in limit case by

$$F_n(f;x) = \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}(k-nx)^2 x^{k-1}(1-x)^{n-k-1} f\left(\frac{k}{n}\right), \tag{4.14}$$

where $f \in C(0,1),\ n \in \mathbb{N}$ and $x \in (0,1)$.

In our research, we generalize the operators defined by Usta in (4.14) with shifted knots as follows:

$$\mathbb{O}_n^{\langle a_i,b_i\rangle}(f;x) = \sum_{k=0}^{n} o_n^{\langle a_i,b_i\rangle}(x)\, f\left(\frac{k+a_1}{n+b_1}\right), \tag{4.15}$$

where

$$o_n^{\langle a_i,b_i\rangle}(x) = (n+b_2)\left(\frac{n+b_2}{n}\right)^{n-1}\left(\frac{k+a_2}{n+b_2}-x\right)^2\left(x-\frac{a_2}{n+b_2}\right)^{k-1}\left(\frac{n+a_2}{n+b_2}-x\right)^{n-k-1},$$

$x \in J_n = \left(\frac{a_2}{n+b_2}, \frac{n+a_2}{n+b_2}\right)$, and $a_p,\ b_p,\ p = 1,2$ are positive real numbers with the condition $0 \leqslant a_2 \leqslant a_1 \leqslant b_1 \leqslant b_2$. The convergence properties of these operators in a moving interval that expands to $(0,1)$ were the main focus of this article. It is worth noting that for n, the variable x is placed in the interval J_n, and as n approaches infinity, the interval J_n turns out to be interval $(0,1)$. These operators clearly include operators (4.14) for $a_1 = a_2 = b_1 = b_2 = 0$.

4.2.2 Preliminaries

Lemma 4.2.1 *The following equalities hold for the proposed operators $\mathbb{O}_n^{\langle a_i,b_i\rangle}$ described by equation (4.15):*

$$\mathbb{O}_n^{\langle a_i,b_i\rangle}(1;x) = 1;$$

$$\mathbb{O}_n^{\langle a_i,b_i\rangle}(t;x) = \left(1-\frac{2}{n}\right)\left(\frac{n+b_2}{n+b_1}\right)x + \frac{(a_1-a_2+1)n+2a_2}{n(n+b_1)};$$

$$\mathbb{O}_n^{\langle a_i,b_i\rangle}\left(t^2;x\right)=\left(1-\frac{6}{n}\right)\left(1-\frac{1}{n}\right)\left(\frac{n+b_2}{n+b_1}\right)^2x^2$$
$$+\frac{\left(2a_1n^2-2a_2n^2-4a_1n+14a_2n-12a_2+5n^2-6n\right)(n+b_2)}{n^2(n+b_1)^2}x$$
$$+\frac{a_1^2n^2+a_2^2n^2+2a_1n^2-2a_1a_2n^2-5a_2n^2-7a_2^2n+4a_1a_2n+6a_2n+6a_2^2+n^2}{n^2(n+b_1)^2}.$$

Lemma 4.2.2 *For the operators* $\mathbb{O}_n^{\langle a_i,b_i\rangle}$, *we have*

$$\lim_{n\to\infty} n\mathbb{O}_n^{\langle a_i,b_i\rangle}\left((t-x);x\right)=1-2x+(a_1-a_2)-x\left(b_1-b_2\right);$$

$$\lim_{n\to\infty} n\mathbb{O}_n^{\langle a_i,b_i\rangle}\left((t-x)^2;x\right)=3\left(1-x\right)x;$$

$$\lim_{n\to\infty} n^2\mathbb{O}_n^{\langle a_i,b_i\rangle}\left((t-x)^4;x\right)=15(1-x)^2x^2.$$

4.2.3 Convergence Properties of $\mathbb{O}_n^{\langle a_i,b_i\rangle}$

The following theorem is a fundamental result on convergence of $\mathbb{O}_n^{\langle a_i,b_i\rangle}$ to $f(x)$:

Theorem 4.2.3 *For a continuous function on* $(0,1)$, *the equivalence*

$$\lim_{n\to\infty}\max_{\frac{a_2}{n+b_2}<x<\frac{n+a_2}{n+b_2}}\left|\mathbb{O}_n^{\langle a_i,b_i\rangle}\left(f;x\right)-f(x)\right|=0,$$

is valid.

Proof: Consider the sequence of operators

$$\mathbb{O}_n^{\langle a_i,b_i\rangle}\left(f;x\right)=\begin{cases}\mathbb{O}_n^{\langle a_i,b_i\rangle}\left(f;x\right), & if\ \ \dfrac{a_2}{n+b_2}<x<\dfrac{n+a_2}{n+b_2},\\ f\left(x\right), & if\ \ x\in\left(0,\dfrac{a_2}{n+b_2}\right]\cup\left[\dfrac{n+a_2}{n+b_2},1\right)\end{cases}.$$

Then,

$$\left\|\mathbb{O}_n^{\langle a_i,b_i\rangle}\left(f;x\right)-f\left(x\right)\right\|=\max_{\frac{a_2}{n+b_2}<x<\frac{n+a_2}{n+b_2}}\left|\mathbb{O}_n^{\langle a_i,b_i\rangle}\left(f;x\right)-f\left(x\right)\right|. \tag{4.16}$$

In view of Lemma 4.2.1, for $p=0,1,2$, we can write

$$\lim_{n\to\infty}\max_{\frac{a_2}{n+b_2}<x<\frac{n+a_2}{n+b_2}}\left|\mathbb{O}_n^{\langle a_i,b_i\rangle}\left(t^p;x\right)-x^p\right|=0.$$

Using above relation, we get

$$\lim_{n\to\infty}\left\|\mathbb{O}_n^{\langle a_i,b_i\rangle}\left(t^p;x\right)-x^p\right\|=0,\qquad p=0,1,2.$$

It is clear from applying Korovkin's theorem [118] to the sequence of positive linear operators $\mathbb{O}_n^{\langle a_i,b_i\rangle}$, that

$$\lim_{n\to\infty}\left\|\mathbb{O}_n^{\langle a_i,b_i\rangle}(f;x)-f(x)\right\|=0,$$

for every $f\in C(0,1)$. Hence equation (4.16) gives

$$\lim_{n\to\infty}\max_{\frac{a_2}{n+b_2}<x<\frac{n+a_2}{n+b_2}}\left|\mathbb{O}_n^{\langle a_i,b_i\rangle}(f;x)-f(x)\right|=0,$$

which is the required result.

Moduli of continuity of various forms are crucial tools in approximation theory for measuring the degree of approximation. The convergence rate of the sequence $\mathbb{O}_n^{\langle a_i,b_i\rangle}$ to the function $f(x)$ by modulus of continuity is given in the next theorem:

Theorem 4.2.4 *For a function $f\in C(0,1)$ and $x\in\left(\frac{a_2}{n+b_2},\frac{n+a_2}{n+b_2}\right)$, the inequality*

$$\left|\mathbb{O}_n^{\langle a_i,b_i\rangle}(f;x)-f(x)\right|\leqslant 2\omega\left(f;\sqrt{C_{n,2}^{\langle a_i,b_i\rangle}(x)}\right)$$

holds true.

Proof: As a result of positivty and linearity of the operators and utilizing property $|f(t)-f(x)|\leqslant\left(1+\frac{(t-x)^2}{\delta^2}\right)\omega(f;\delta)$ of modulus of continuity, we have

$$\begin{aligned}\left|\mathbb{O}_n^{\langle a_i,b_i\rangle}(f;x)-f(x)\right|&\leqslant\mathbb{O}_n^{\langle a_i,b_i\rangle}(|f(t)-f(x)|;x)\\&\leqslant\left(1+\frac{\mathbb{O}_n^{\langle a_i,b_i\rangle}\left((t-x)^2;x\right)}{\delta^2}\right)\omega(f;\delta).\end{aligned}$$

Choosing $\delta^2=C_{n,2}^{\langle a_i,b_i\rangle}(x)=\mathbb{O}_n^{\langle a_i,b_i\rangle}\left((t-x)^2;x\right)$, the desired result is simply achieved.

Theorem 4.2.5 *For every $x\in\left(\frac{a_2}{n+b_2},\frac{n+a_2}{n+b_2}\right)$ and $f\in C(0,1)$ there exists a positive constant C, such that the inequality*

$$\left|\mathbb{O}_n^{\langle a_i,b_i\rangle}(f;x)-f(x)\right|\leqslant C\omega_2\left(f;\frac{1}{2}\sqrt{\Upsilon_n^{\langle a_i,b_i\rangle}(x)}\right)+\omega\left(f;|(\phi(n)-1)x+\psi(n)|\right),$$

holds true. Where $\Upsilon_n^{\langle a_i,b_i\rangle}(x)=C_{n,2}^{\langle a_i,b_i\rangle}(x)+((\phi(n)-1)x+\psi(n))^2$, $\phi(n)=\left(1-\frac{2}{n}\right)\left(\frac{n+b_2}{n+b_1}\right)$ and $\psi(n)=\frac{(a_1-a_2+1)n+2a_2}{n(n+b_1)}$.

Proof: We begin our proof by considering the following auxiliary operators:

$$\widehat{O}_n^{\langle a_i,b_i\rangle}(f;x)=\mathbb{O}_n^{\langle a_i,b_i\rangle}(f;x)-f(\phi(n)x+\psi(n))+f(x),\tag{4.17}$$

where $f\in C(0,1)$, $\phi(n)=\left(1-\frac{2}{n}\right)\left(\frac{n+b_2}{n+b_1}\right)$ and $\psi(n)=\frac{(a_1-a_2+1)n+2a_2}{n(n+b_1)}$. From Lemma 4.2.1, we find

$$\widehat{O}_n^{\langle a_i,b_i\rangle}(1;x)=1,\quad\widehat{O}_n^{\langle a_i,b_i\rangle}(t-x;x)=0.$$

For $g \in C^2(0,1) = \{f \in C(0,1) : f'' \in C(0,1)\}$, operating $\widehat{O}_n^{\langle a_i,b_i\rangle}$ on both side of Taylor's expansion, we get

$$\widehat{O}_n^{\langle a_i,b_i\rangle}(f;x) = f(x) + \widehat{O}_n^{\langle a_i,b_i\rangle}((t-x);x)\,g'(x) + \widehat{O}_n^{\langle a_i,b_i\rangle}\left(\int_x^t (t-z)g''(z)dz;x\right)$$
$$= f(x) + \mathbb{O}_n^{\langle a_i,b_i\rangle}\left(\int_x^t (t-z)g''(z)dz;x\right) - \int_x^{\phi(n)x+\psi(n)} (\phi(n)x+\psi(n)-z)g''(z)dz,$$

which implies

$$\left|\widehat{O}_n^{\langle a_i,b_i\rangle}(f;x) - f(x)\right| \leqslant \mathbb{O}_n^{\langle a_i,b_i\rangle}\left(\left|\int_x^t (t-z)g''(z)dz;x\right|\right)$$
$$+\left|\int_x^{\phi(n)x+\psi(n)} |(\phi(n)x+\psi(n)-z)|\,|g''(z)|dz\right|$$
$$\leqslant \mathbb{O}_n^{\langle a_i,b_i\rangle}\left((t-x)^2;x\right)\|g''\| + (\phi(n)x+\psi(n)-x)^2\|g''\|$$
$$= \left\{C_{n,2}^{\langle a_i,b_i\rangle}(x) + (\phi(n)x+\psi(n)-x)^2\right\}\|g''\|$$
$$= \Upsilon_n^{\langle a_i,b_i\rangle}(x)\,\|g''\|. \tag{4.18}$$

From equation (4.17), we have

$$\left|\widehat{O}_n^{\langle a_i,b_i\rangle}(f;x)\right| \leqslant \left|\mathbb{O}_n^{\langle a_i,b_i\rangle}(f;x)\right| + |f(x)| + |f(\phi(n)x+\psi(n))|$$
$$\leqslant \|f\|\,\mathbb{O}_n^{\langle a_i,b_i\rangle}(1;x) + 2\|f\| = 3\|f\|. \tag{4.19}$$

Based on the concept of modulus of continuity and equations (4.18), (4.19), we have

$$\left|\mathbb{O}_n^{\langle a_i,b_i\rangle}(f;x) - f(x)\right|$$
$$= \left|\widehat{O}_n^{\langle a_i,b_i\rangle}(f;x) - f(x) + f(\phi(n)x+\psi(n)) - f(x)\right|$$
$$\leqslant \left|\widehat{O}_n^{\langle a_i,b_i\rangle}(f-g;x)\right| + \left|\widehat{O}_n^{\langle a_i,b_i\rangle}(g;x) - g(x)\right| + |f(x)-g(x)| + |f(\phi(n)x+\psi(n)) - f(x)|$$
$$\leqslant 4\|f-g\| + \Upsilon_n^{\langle a_i,b_i\rangle}(x)\,\|g''\| + \omega(f;|\phi(n)x+\psi(n)-x|).$$

Taking the infimum over all $g \in C^2(0,1)$ and using Peetre's K-functional, we get

$$\left|\mathbb{O}_n^{\langle a_i,b_i\rangle}(f;x) - f(x)\right| \leqslant 4K\left(f;\frac{1}{4}\Upsilon_n^{\langle a_i,b_i\rangle}(x)\right) + \omega(f;|(\phi(n)-1)x+\psi(n)|).$$

Hence, proof is completed by using relation (1.2).

4.2.4 Weighted Approximation

In this section, we refer to subsection 1.1.5 with the interval $I = (0,1)$ and offer the theorems of the Korovkin type for the weighted approximation of newly constructed operators $\mathbb{O}_n^{\langle a_i,b_i\rangle}$. In order to achieve this goal, we follow the theorems stated by Gadjiev in [81]. For all $x \in (0,1)$, set $\rho(x) = 1 + x^2$ as a continuous weight function on $(0,1)$ and $\lim\limits_{|x|\to\infty} \rho(x) = \infty,\ \rho(x) \geqslant 1$. The primary theorem of this section can now be presented as follows.

Lemma 4.2.6 *For a function $f \in C_\rho(0,1)$, the inequality*

$$\left\|\mathbb{O}_n^{\langle a_i,b_i\rangle}(f)\right\|_\rho \leqslant C\|f\|_\rho,$$

holds true for the operators $\mathbb{O}_n^{\langle a_i,b_i\rangle}$.

Proof: The demonstration of this lemma clearly follows in view of the formulation of newly constructed operators $\mathbb{O}_n^{\langle a_i,b_i\rangle}$ and the properties of preserving polynomials.

Theorem 4.2.7 *Let $f \in C_\rho(0,1)$. Then, for the operators $\mathbb{O}_n^{\langle a_i,b_i\rangle}$, the equivalence*

$$\lim_{n\to\infty}\left\|\mathbb{O}_n^{\langle a_i,b_i\rangle}(f) - f\right\|_\rho = 0$$

holds.

Proof: In order to prove this theorem, it is sufficient to demonstrate that the following three conditions are true:

$$\lim_{n\to\infty}\left\|\mathbb{O}_n^{\langle a_i,b_i\rangle}(t^r) - x^r\right\|_\rho = 0, \qquad r = 0,1,2$$

The fact that $\mathbb{O}_n^{\langle a_i,b_i\rangle}(1;x) = 1$, makes the first condition evident to hold true. By applying the relevant consequences of Lemma 4.2.1 for the remaining ones, we get

$$\begin{aligned}
&\left\|\mathbb{O}_n^{\langle a_i,b_i\rangle}(t) - x\right\|_\rho \\
&= \sup_{x\in(0,1)} \frac{\left|\mathbb{O}_n^{\langle a_i,b_i\rangle}(t;x) - x\right|}{1+x^2} \\
&\leqslant \left|\left(1-\frac{2}{n}\right)\left(\frac{n+b_2}{n+b_1}\right) - 1\right| \sup_{x\in(0,1)} \frac{x}{1+x^2} + \left|\frac{(a_1-a_2+1)\,n+2a_2}{n\,(n+b_1)}\right| \sup_{x\in(0,1)} \frac{1}{1+x^2} \\
&= |A(n)| \sup_{x\in(0,1)} \frac{x}{1+x^2} + |B(n)| \sup_{x\in(0,1)} \frac{1}{1+x^2}.
\end{aligned}$$

The second condition is supported by the fact that $\lim_{n\to\infty}|A(n)| = 0$ and $\lim_{n\to\infty}|B(n)| = 0$. Similarly,

$$\left\|\mathbb{O}_n^{\langle a_i,b_i\rangle}\left(t^2\right)-x^2\right\|_\rho$$
$$= \sup_{x\in(0,1)} \frac{\left|\mathbb{O}_n^{\langle a_i,b_i\rangle}\left(t^2;x\right)-x^2\right|}{1+x^2}$$
$$\leqslant \left|\left(1-\frac{6}{n}\right)\left(1-\frac{1}{n}\right)\left(\frac{n+b_2}{n+b_1}\right)^2-1\right| \sup_{x\in(0,1)}\frac{x^2}{1+x^2}$$
$$+\left|\frac{\left(2a_1n^2-2a_2n^2-4a_1n+14a_2n-12a_2+5n^2-6n\right)(n+b_2)}{n^2(n+b_1)^2}\right| \sup_{x\in(0,1)}\frac{x}{1+x^2}$$
$$+\left|\frac{a_1^2n^2+a_2^2n^2+2a_1n^2-2a_1a_2n^2-5a_2n^2-7a_2^2n+4a_1a_2n+6a_2n+6a_2^2+n^2}{n^2(n+b_1)^2}\right| \sup_{x\in(0,1)}\frac{1}{1+x^2}$$
$$= |C(n)| \sup_{x\in(0,1)}\frac{x^2}{1+x^2}+|D(n)| \sup_{x\in(0,1)}\frac{x}{1+x^2}+|E(n)| \sup_{x\in(0,1)}\frac{1}{1+x^2}.$$

This indicates that the third condition is satisfied because $\lim_{n\to\infty}|C(n)|$, $\lim_{n\to\infty}|D(n)|$ and $\lim_{n\to\infty}|E(n)|$ is equals to zero. Hence, the proof is completed.

4.2.5 Voronovskaja Theorem

Let $C^2(0,1)$ be the space containing all functions f such that $f'' \in C(0,1)$. The following theorem proves a quantitative Voronovskaja type theorem for the operators $\mathbb{O}_n^{\langle a_i,b_i\rangle}$ using the Ditzian-Totik modulus of smoothness defined in subsection 1.1.4.

Theorem 4.2.8 *Let $g \in C^2(0,1)$ and sufficiently large n, the following inequality*

$$\left|\mathbb{O}_n^{\langle a_i,b_i\rangle}(f;x)-f(x)-f'(x)C_{n,1}^{\langle a_i,b_i\rangle}(x)-f''(x)C_{n,2}^{\langle a_i,b_i\rangle}(x)\right| \leqslant \frac{1}{n}C\varphi^2(x)\omega_\varphi(f'';n^{-\frac{1}{2}}),$$

holds true. Where $C_{n,1}^{\langle a_i,b_i\rangle}(x) = \mathbb{O}_n^{\langle a_i,b_i\rangle}((t-x);x)$, $C_{n,2}^{\langle a_i,b_i\rangle}(x) = \mathbb{O}_n^{\langle a_i,b_i\rangle}\left((t-x)^2;x\right)$ and C is a positive constant.

Proof: For $g \in C^2(0,1), t, x \in (0,1)$, by Taylor's expansion, we have

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t f''(y)(t-y)dy.$$

Hence

$$f(t)-f(x)-f'(x)(t-x)-\frac{1}{2}f''(x)(t-x)^2 = \int_x^t f''(y)(t-y)dy - \int_x^t f''(x)(t-y)dy$$
$$= \int_x^t \left[f''(y)-f''(x)\right](t-y)dy.$$

Applying $\mathbb{O}_n^{\langle a_i,b_i\rangle}$ to both sides of the above relation, we get

$$\begin{aligned}&\left|\mathbb{O}_n^{\langle a_i,b_i\rangle}(f;x)-f(x)-f'(x)C_{n,1}^{\langle a_i,b_i\rangle}(x)-f''(x)C_{n,2}^{\langle a_i,b_i\rangle}(x)\right|\\&\leqslant \mathbb{O}_n^{\langle a_i,b_i\rangle}\Big(\left|\int_x^t |f''(y)-f''(x)|\,|t-y|dy\right|;x\Big).\end{aligned}\tag{4.20}$$

In [79], the quantity $\left|\int_x^t |f''(y)-f''(x)|\,|t-y|dy\right|$ was estimated as follows:

$$\left|\int_x^t |f''(y)-f''(x)|\,|t-y|dy\right|\leqslant 2\,\|f''-h\|\,(t-x)^2+2\,\|\varphi h'\|\,\varphi^{-1}(x)|t-x|^3,\tag{4.21}$$

where $f\in W_\varphi(0,1)$.
Using Lemma 4.2.2, it follows that for sufficiently large n, a constant $C>0$ exists, such that

$$\mathbb{O}_n^{\langle a_i,b_i\rangle}((t-x)^2;x)\leqslant \frac{C}{n}\varphi^2(x)\ \ and\ \ \mathbb{O}_n^{\langle a_i,b_i\rangle}((t-x)^4;x)\leqslant \frac{C}{n^2}\varphi^4(x).\tag{4.22}$$

From (4.20)-(4.22) and applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\left|\mathbb{O}_n^{\langle a_i,b_i\rangle}(f;x)-f(x)-f'(x)C_{n,1}^{\langle a_i,b_i\rangle}(x)-f''(x)C_{n,2}^{\langle a_i,b_i\rangle}(x)\right|\\&\leqslant 2\,\|f''-h\|\,\mathbb{O}_n^{\langle a_i,b_i\rangle}((t-x)^2;x)+2\,\|\varphi h'\|\,\varphi^{-1}(x)\mathbb{O}_n^{\langle a_i,b_i\rangle}\left(|t-x|^3;x\right)\\&\leqslant \frac{C}{n}\varphi^2(x)\,\|f''-h\|+2\,\|\varphi h'\|\,\varphi^{-1}(x)\left\{\mathbb{O}_n^{\langle a_i,b_i\rangle}|t-x|^2;x\right\}^{\frac12}\left\{\mathbb{O}_n^{\langle a_i,b_i\rangle}|t-x|^4;x\right\}^{\frac12}\\&\leqslant \frac{C}{n}\varphi^2(x)\,\|f''-h\|+\varphi^2(x)\frac{C}{n\sqrt{n}}\,\|\varphi h'\|\leqslant \frac{C}{n}\varphi^2(x)\left(\|f''-h\|+n^{-\frac12}\,\|\varphi h'\|\right).\end{aligned}$$

The theorem is proved by taking the infimum on the right-hand side of the above relation over $f\in W_\varphi(0,1)$.

Corollary 4.2.9 *If $g\in C^2(0,1)$, then*

$$\lim_{n\to\infty} n\left\{\mathbb{O}_n^{\langle a_i,b_i\rangle}(f;x)-f(x)-f'(x)C_{n,1}^{\langle a_i,b_i\rangle}(x)-f''(x)C_{n,2}^{\langle a_i,b_i\rangle}(x)\right\}=0,$$

where $C_{n,1}^{\langle a_i,b_i\rangle}(x)$ and $C_{n,2}^{\langle a_i,b_i\rangle}(x)$ are defined in above Theorem.

In [16], a Grüss inequality for positive linear operators was established by using the least concave majorant of modulus of continuity. This result sparked a considerable lot of interest after its publication. For a class of sequences of positive linear operators, Acar et al. [9] proved a Grüss-Voronovskaja type theorem and a Grüss type approximation theorem. The following result is the Grüss-Voronovskaja type theorem for proposed operators.

Theorem 4.2.10 *Let $f, g \in C^2(0,1)$. Then, for each $x \in \left(\frac{a_2}{n+b_2}, \frac{n+a_2}{n+b_2}\right)$*

$$\lim_{n\to\infty} n\left\{\mathbb{O}_n^{\langle a_i,b_i\rangle}(fg;x) - \mathbb{O}_n^{\langle a_i,b_i\rangle}(f;x)\mathbb{O}_n^{\langle a_i,b_i\rangle}(g;x)\right\} = 6f'(x)g'(x)(1-x)x.$$

Proof: We begin our proof with the following relation:

$$\begin{aligned}
&\mathbb{O}_n^{\langle a_i,b_i\rangle}((fg);x) - \mathbb{O}_n^{\langle a_i,b_i\rangle}(f;x)\,\mathbb{O}_n^{\langle a_i,b_i\rangle}(g;x)\\
&= \mathbb{O}_n^{\langle a_i,b_i\rangle}((fg);x) - f(x)\,g(x) - (fg)'(x)C_{n,1}^{\langle a_i,b_i\rangle}(x) - (fg)''(x)C_{n,2}^{\langle a_i,b_i\rangle}(x)\\
&- \mathbb{O}_n^{\langle a_i,b_i\rangle}(f;x)\left\{\mathbb{O}_n^{\langle a_i,b_i\rangle}(g;x) - g(x) - g'(x)C_{n,1}^{\langle a_i,b_i\rangle}(x) - g''(x)C_{n,2}^{\langle a_i,b_i\rangle}(x)\right\}\\
&- g(x)\left\{\mathbb{O}_n^{\langle a_i,b_i\rangle}(f;x) - f(x) - f'(x)C_{n,1}^{\langle a_i,b_i\rangle}(x) - f''(x)C_{n,2}^{\langle a_i,b_i\rangle}(x)\right\}\\
&+ C_{n,2}^{\langle a_i,b_i\rangle}(x)\left\{f(x)g''(x) + 2f'(x)g'(x) - g''(x)\mathbb{O}_n^{\langle a_i,b_i\rangle}(f;x)\right\}\\
&+ C_{n,1}^{\langle a_i,b_i\rangle}(x)\left\{f(x)g'(x) - g'(x)\mathbb{O}_n^{\langle a_i,b_i\rangle}(f;x)\right\}.
\end{aligned}$$

Now, using Corollary 4.2.9 and Lemma 4.2.2, we get

$$\begin{aligned}
&\lim_{n\to\infty} n\left\{\mathbb{O}_n^{\langle a_i,b_i\rangle}((fg);x) - \mathbb{O}_n^{\langle a_i,b_i\rangle}(f;x)\,\mathbb{O}_n^{\langle a_i,b_i\rangle}(g;x)\right\}\\
&= \lim_{n\to\infty} 2nf'(x)g'(x)C_{n,2}^{\langle a_i,b_i\rangle}(x) + \lim_{x\to\infty} ng''(x)\left\{f(x) - \mathbb{O}_n^{\langle a_i,b_i\rangle}(f;x)\right\}C_{n,2}^{\langle a_i,b_i\rangle}(x)\\
&+ \lim_{n\to\infty} ng'(x)\left\{f(x) - \mathbb{O}_n^{\langle a_i,b_i\rangle}(f;x)\right\}C_{n,1}^{\langle a_i,b_i\rangle}(x) = 6f'(x)g'(x)(1-x)\,x.
\end{aligned}$$

4.2.6 Numerical Results

Example 4.2.11 *The convergence of $\mathbb{O}_{100}^{\langle a_i,b_i\rangle}$ (magenta), $\mathbb{O}_{200}^{\langle a_i,b_i\rangle}$ (red) and $\mathbb{O}_{400}^{\langle a_i,b_i\rangle}$ (blue) to $f(x) = 2\sin\left(\frac{\pi x}{2}\right) + \cos^3\left(2\pi x - \frac{\pi}{2}\right)$ (black) is illustrated in Figure 4.7 for fixed $a_1 = 2$, $a_2 = 1$, $b_1 = 3$ and $b_2 = 4$. Table 4.2 computes the absolute error $\mathcal{E}_n^{\langle a_i,b_i\rangle}(x) = \left|\mathbb{O}_n^{\langle a_i,b_i\rangle}(f;x) - f(x)\right|$ of the function f for various values of x in the interval $(0,1)$, and Figure 4.8 displays this error graphically. When n rises from 100 to 400, we notice that the approximation of f by $\mathbb{O}_n^{\langle a_i,b_i\rangle}$ gets better and error also continues to decrease.*

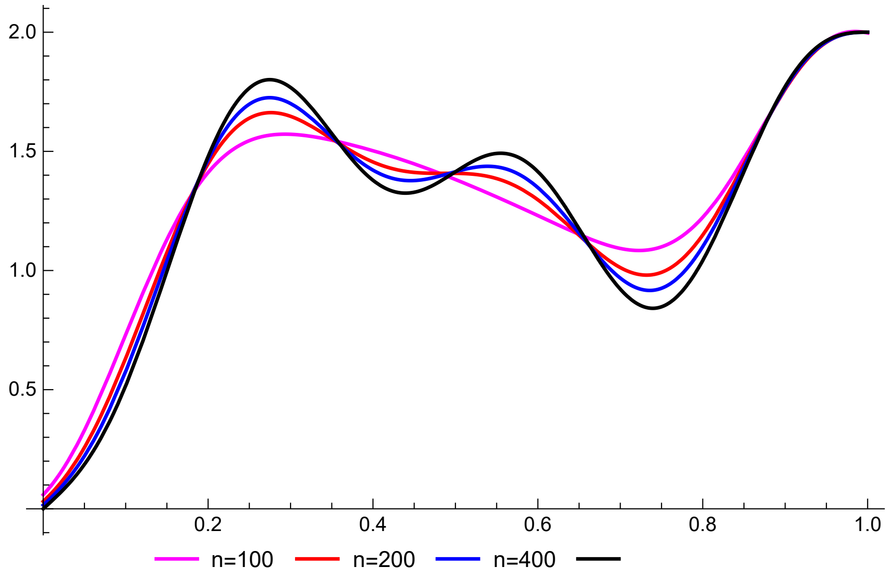


Figure 4.7: Convergence of $\mathbb{O}_{100}^{\langle a_i,b_i\rangle}$ (magenta), $\mathbb{O}_{200}^{\langle a_i,b_i\rangle}$ (red) and $\mathbb{O}_{400}^{\langle a_i,b_i\rangle}$ (blue) for fixed $a_1 = 2,\ a_2 = 1,\ b_1 = 3$ and $b_2 = 4$ to $f(x) = 2\sin\left(\frac{\pi x}{2}\right) + \cos^3\left(2\pi x - \frac{\pi}{2}\right)$ (black).

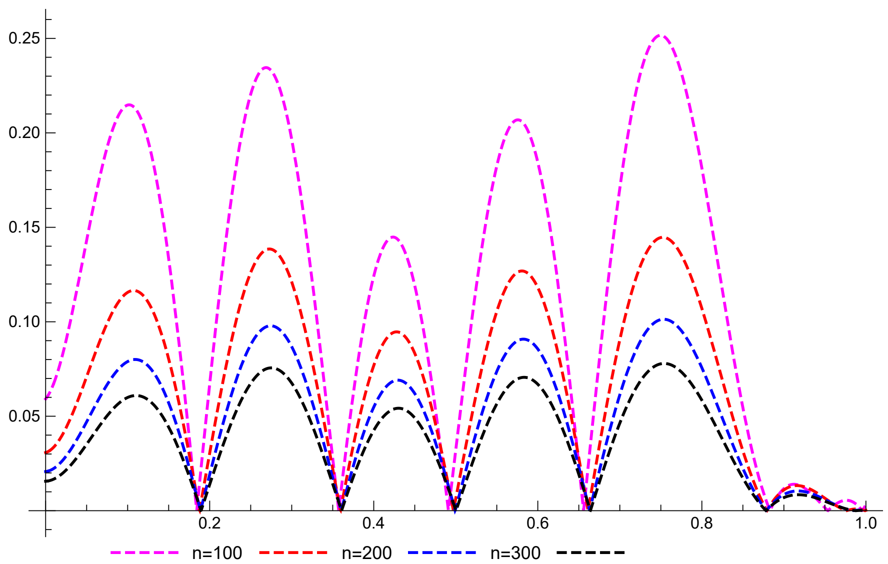


Figure 4.8: Graph of $\mathcal{E}_{100}^{\langle a_i,b_i\rangle}(x)$ (magenta), $\mathcal{E}_{200}^{\langle a_i,b_i\rangle}(x)$ (red), $\mathcal{E}_{300}^{\langle a_i,b_i\rangle}(x)$ (blue) and $\mathcal{E}_{400}^{\langle a_i,b_i\rangle}(x)$ (black) for $a_1 = 2,\ a_2 = 1,\ b_1 = 3$ and $b_2 = 4$ to $f(x) = 2\sin\left(\frac{\pi x}{2}\right) + \cos^3\left(2\pi x - \frac{\pi}{2}\right)$.

Table 4.2: Estimation of error for various value of x in the interval $(0,1)$

x	$\mathcal{E}_{100}^{\langle a_i,b_i\rangle}(x)$	$\mathcal{E}_{200}^{\langle a_i,b_i\rangle}(x)$	$\mathcal{E}_{300}^{\langle a_i,b_i\rangle}(x)$	$\mathcal{E}_{400}^{\langle a_i,b_i\rangle}(x)$
0.1	0.214722	0.115569	0.0789526	0.059939
0.2	0.0670277	0.0304526	0.0191637	0.0138667
0.3	0.196595	0.121108	0.0868044	0.0675388
0.4	0.124112	0.0767954	0.0548047	0.0425002
0.5	0.0321965	0.00647151	0.00125015	0.000373294
0.6	0.184441	0.118125	0.0855918	0.0669356
0.7	0.170772	0.0905297	0.0613788	0.0463937
0.8	0.181367	0.10707	0.0756795	0.0584801
0.9	0.0116926	0.0109919	0.00857444	0.00691517

Chapter 5

Bivariate generalization for operators involving Appostol-Genocchi polynomial

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The late 19*th* *century witnessed the pioneering contributions of Chebyshev to the field of orthogonal polynomials, which were subsequently expanded upon by Markov and Steiltjes. This chapter is primarily concerned with the bivariate generalization of operators involving a class of orthogonal polynomials called Apostol-Genocchi polynomials. The rate of convergence can be determined in terms of partial and total modulus of continuity as well as the order of approximation can be achieved by means of a Lipschitz-type function and Peetre's K-functional. In addition, we put forth a conceptual extension known as the "generalized boolean sum (GBS)" for these bivariate operators, which aims to establish the degree of approximation for Bögel continuous functions. In this study, we utilize the Mathematica Software to present a series of graphical illustrations that effectively showcase the rate of convergence for the bivariate operators. The graphs indicate that, in the case of certain functions, the bivariate operators exhibits superior convergence when* α *is less than* β*. Based on our analysis and comparison of the error of approximation between the bivariate operators and the corresponding GBS operators, it can be deduced that the GBS operators exhibit a faster convergence towards the function.*

5.1 Introduction

The Genocchi polynomials $g_k(x)$ are generally proposed by means of the following generating functions

$$\frac{2te^{xt}}{e^t+1} = \sum_{k=0}^{\infty} g_k(x) \frac{t^k}{k!}.$$

In particular, $g_k := g_k(0)$ for $n \geq 0$ are called Genocchi numbers with $g_{2k+1} = 0$ for $k \geq 1$. Luo [124], extended the Genocchi polynomial in x. These extended polynomials are called Apostol-Genocchi polynomials $g_k(x;\lambda)$ and given by the means of the following generating functions:

$$\frac{2te^{xt}}{\lambda e^t+1} = \sum_{k=0}^{\infty} g_k(x;\lambda) \frac{t^k}{k!} \left(|t| < |\log(-\lambda)|\right).$$

The Apostol-Genocchi polynomials [108] $g_k^{(a)}(x;\lambda)$ of order α in the variable x are proposed using the generating function defined as:

$$\left(\frac{2t}{1+\beta e^t}\right)^{\alpha} e^{xt} = \sum_{k=0}^{\infty} g_k^{\alpha}(x;\beta) \frac{t^k}{k!}, \qquad (|t| < \pi). \tag{5.1}$$

For $f \in C[0,\infty)$, Prakash et al. [143] considered the following operators:

$$G_n^{\alpha,\beta}(f;x) = e^{-nx}\left(\frac{1+e\beta}{2}\right)^{\alpha} \sum_{k=0}^{\infty} \frac{g_k^{\alpha}(nx;\beta)}{k!} f\left(\frac{k}{n}\right), \tag{5.2}$$

where $g_k^{\alpha}(x;\beta)$ is generalized Apostol-Genocchi polynomials with generating function given in (5.1). The Apostol-Genocchi polynomials and their associated properties are studied by many researchers. For further information, readers can refer (cf. [124; 125; 139; 156]).

In [125], the following explicit series representation for the Apostol-Genocchi polynomials is given as:

$$g_k^{\alpha}(x;\beta) = 2^{\alpha}\alpha!\binom{k}{\alpha}\sum_{i=0}^{k-\alpha}\frac{\beta^i}{(1+\beta)^{\alpha+i}}\binom{k-\alpha}{i}\binom{\alpha+i-1}{i}$$
$$\times \sum_{j=0}^{i}(-1)^j\binom{i}{j}j^i(x+j)^{k-i-\alpha}{}_2F_1[\alpha+i-k, i; i+1; j/(x+j)],$$

where $k,\alpha \in \mathbb{N}\cup\{0\}$, $\beta \in \mathbb{R}\backslash\{-1\}$ and ${}_2F_1[a,b;c;x]$ indicates the Gaussian hypergeometric function given by

$${}_2F_1[a,b;c;x] = {}_2F_1[b,a;c;x] = \sum_{j=0}^{\infty}\frac{(a)_j(b)_j}{(c)_j}\frac{x^j}{j!},$$

where $(a)_0 = 1$, $(a)_j = a(a+1)\cdots(a+j-1) = \frac{\Gamma(j+a)}{\Gamma(a)}$.

Deo and kumar [64] introduced Durrmeyer variant of operators (5.2) and gave approximation results. In this study, we aim to investigate the bivariate generalization of the Apostol-Genocchi operators defined in equation (5.2).

5.2 Construction of Bivariate Operators

Consider the interval $I = [0,\infty] \times [0,\infty]$. Let $C(I)$ represent the set of all real valued continuous functions on interval I. For a function $f \in C(I)$, the bivariate generalization of the operators (5.2) is defined as:

$$G_{n_1,n_2}^{\alpha,\beta}(f;x_1,x_2) = e^{-(n_1x_1+n_2x_2)}\left(\frac{1+e\beta}{2}\right)^{2\alpha}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{g_{k_1}^{\alpha}(n_1x_1;\beta)}{k_1!}\frac{g_{k_2}^{\alpha}(n_2x_2;\beta)}{k_2!}f\left(\frac{k_1}{n_1},\frac{k_2}{n_2}\right), \tag{5.3}$$

Some preliminary concepts for the above-mentioned operators that are useful throughout the paper are outlined below:

Lemma 5.2.1 *Let $e_{rs}(t_1,t_2) = t_1^r t_2^s$, $0 \le r+s \le 2$. For $(x_1,x_2) \in I$, we have*

$$\begin{aligned}
G_{n_1,n_2}^{\alpha,\beta}(1;x_1,x_2) &= 1;\\
G_{n_1,n_2}^{\alpha,\beta}(t_1;x_1,x_2) &= x_1 + \frac{\alpha}{n_1(1+e\beta)};\\
G_{n_1,n_2}^{\alpha,\beta}(t_2;x_1,x_2) &= x_2 + \frac{\alpha}{n_2(1+e\beta)};\\
G_{n_1,n_2}^{\alpha,\beta}\left(t_1^2;x_1,x_2\right) &= x_1^2 + \frac{(1+2\alpha+e\beta)}{n_1(1+e\beta)}x_1 + \frac{\alpha^2-2\alpha e\beta-\alpha e^2\beta^2}{{n_1}^2(1+e\beta)^2};\\
G_{n_1,n_2}^{\alpha,\beta}\left(t_2^2;x_1,x_2\right) &= x_2^2 + \frac{(1+2\alpha+e\beta)}{n_2(1+e\beta)}x_2 + \frac{\alpha^2-2\alpha e\beta-\alpha e^2\beta^2}{{n_2}^2(1+e\beta)^2}.
\end{aligned}$$

Remark 5.2.2 *Using lemma 5.2.1 , we have*

$$\begin{aligned}
G_{n_1,n_2}^{\alpha,\beta}((t_1-x_1);x_1,x_2) &= \frac{\alpha}{n_1(1+e\beta)};\\
G_{n_1,n_2}^{\alpha,\beta}((t_2-x_2);x_1,x_2) &= \frac{\alpha}{n_2(1+e\beta)};\\
G_{n_1,n_2}^{\alpha,\beta}\left((t_1-x_1)^2;x_1,x_2\right) &= \frac{x_1}{n_1} + \frac{\alpha^2-2\alpha e\beta-\alpha e^2\beta^2}{{n_1}^2(1+e\beta)^2};\\
G_{n_1,n_2}^{\alpha,\beta}\left((t_2-x_2)^2;x_1,x_2\right) &= \frac{x_2}{n_2} + \frac{\alpha^2-2\alpha e\beta-\alpha e^2\beta^2}{{n_2}^2(1+e\beta)^2}.
\end{aligned}$$

Furthermore,

$$\lim_{n_1\to\infty} n_1 G_{n_1,n_2}^{\alpha,\beta}((t_1-x_1);x_1) = \frac{\alpha}{(1+e\beta)} \text{ and } \lim_{n_2\to\infty} n_2 G_{n_1,n_2}^{\alpha,\beta}(t_2-x_2);x_2) = \frac{\alpha}{(1+e\beta)},$$
$$\lim_{n_1\to\infty} n_1(G_{n_1,n_2}^{\alpha,\beta}(t_1-x_1)^2;x_1) = x_1 \text{ and } \lim_{n_2\to\infty} n_2(G_{n_1,n_2}^{\alpha,\beta}(t_2-x_2)^2;x_2) = x_2.$$

5.3 Main Results

We determine the order of approximation of operators (5.3) in the space of continuous functions on the set $I_{\mu\kappa} = [0,\mu] \times [0,\kappa]$ in terms of total modulus of continuity and partial modulus of continuity defined in subsection 1.1.8.

Theorem 5.3.1 *If $f \in C(I)$, then*

$$\lim_{(n_1,n_2)\to\infty} G^{\alpha,\beta}_{n_1,n_2}(f; x_1, x_2) = f(x_1, x_2).$$

the above convergence is uniform in each compact $I_{\mu\kappa}$.

Proof: The proof follows from the Volkov Theorem [169].

Theorem 5.3.2 *Let $f \in C(I)$, then for all $(x_1, x_2) \in I$, the inequality*

(i) $\left|G^{\alpha,\beta}_{n_1,n_2}(f; x_1, x_2) - f(x_1, x_2)\right| \leq 2\left(\omega^1(f; \delta_{n_1}) + \omega^2(f; \delta_{n_2})\right),$

(ii) $\left|G^{\alpha,\beta}_{n_1,n_2}(f; x_1, x_2) - f(x_1, x_2)\right| \leq 4\omega(f; \delta_{n_1}, \delta_{n_2}),$

holds true. Where $\delta_{n_1}(x_1) = \sqrt{G^{\alpha,\beta}_{n_1}\left((t_1 - x_1)^2; x_1\right)}$ *and* $\delta_{n_2}(x_2) = \sqrt{G^{\alpha,\beta}_{n_2}\left((t_2 - x_2)^2; x_2\right)}$.

Proof:

(i) Taking into consideration the partial modulus of continuity of $f(x_1, x_2)$ and applying Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\left|G^{\alpha,\beta}_{n_1,n_2}(f; x_1, x_2) - f(x_1, x_2)\right| &\leq G^{\alpha,\beta}_{n_1,n_2}(|f(t_1,t_2) - f(x_1,x_2)|; x_1, x_2) \\
&\leq G^{\alpha,\beta}_{n_1,n_2}(|f(t_1,t_2) - f(x_1,t_2)|; x_1, x_2) \\
&\quad + G^{\alpha,\beta}_{n_1,n_2}(|f(x_1,t_2) - f(x_1,x_2)|; x_1, x_2) \\
&\leq G^{\alpha,\beta}_{n_1,n_2}\left(\omega^1(f; |t_1 - x_1|); x_1, x_2\right) \\
&\quad + G^{\alpha,\beta}_{n_1,n_2}\left(\omega^2(f; |t_2 - x_2|); x_1, x_2\right) \\
&\leq \omega^1(f; \delta_{n_1})\left(1 + \frac{1}{\delta_{n_1}} G^{\alpha,\beta}_{n_1}(|t_1 - x_1|; x_1)\right) \\
&\quad + \omega^2(f; \delta_{n_2})\left(1 + \frac{1}{\delta_{n_2}} G^{\alpha,\beta}_{n_2}(|t_2 - x_2|; x_2)\right) \\
&\leq \omega^1(f; \delta_{n_1})\left(1 + \frac{1}{\delta_{n_1}} G^{\alpha,\beta}_{n_1}\left((t_1 - x_1)^2; x_1\right)^{\frac{1}{2}}\right) \\
&\quad + \omega^2(f; \delta_{n_2})\left(1 + \frac{1}{\delta_{n_2}} G^{\alpha,\beta}_{n_2}\left((t_2 - x_2)^2; x_2\right)^{\frac{1}{2}}\right) \\
&\leq 2\left(\omega^1(f; \delta_{n_1}) + \omega^2(f; \delta_{n_2})\right).
\end{aligned}$$

(ii) Taking into consideration the linearity of $G_{n_1}^{\alpha,\beta}$ and $G_{n_2}^{\alpha,\beta}$, as well as the monotonicity of ω, we can write:

$$\begin{aligned}\left|G_{n_1,n_2}^{\alpha,\beta}(f;x_1,x_2)-f(x_1,x_2)\right| &\le G_{n_1,n_2}^{\alpha,\beta}(|f(t_1,t_2)-f(x_1,x_2)|;x_1,x_2)\\ &\le \omega(f;\delta_{n_1},\delta_{n_2})\left(G_{n_1}^{\alpha,\beta}(1;x_1)+\frac{1}{\delta_{n_1}}G_{n_1}^{\alpha,\beta}(|t_1-x_1|;x_1)\right)\\ &\times\left(G_{n_2}^{\alpha,\beta}(1;x_2)+\frac{1}{\delta_{n_2}}G_{n_2}^{\alpha,\beta}(|t_2-x_2|;x_2)\right).\end{aligned}$$

Applying Cauchy Schwarz inequality, we have

$$\begin{aligned}\left|G_{n_1,n_2}^{\alpha,\beta}(f;x_1,x_2)-f(x_1,x_2)\right| &\le \omega(f;\delta_{n_1},\delta_{n_2})\left(G_{n_1}^{\alpha,\beta}(1;x_1)+\frac{1}{\delta_{n_1}}G_{n_1}^{\alpha,\beta}\Big((t_1-x_1)^2;x_1\Big)^{\frac{1}{2}}\right)\\ &\times\left(G_{n_2}^{\alpha,\beta}(1;x_2)+\frac{1}{\delta_{n_2}}G_{n_2}^{\alpha,\beta}\Big((t_2-x_2)^2;x_2\Big)^{\frac{1}{2}}\right).\end{aligned}$$

Therefore, the desired result is achieved by selecting $\delta_{n_1}(x_1)$ and $\delta_{n_2}(x_2)$ as mentioned in the statement.

Now we determine the order of approximation of bivariate operators (5.3) with the help of Lipschitz class functions. We define Lipschitz class $Lip_{\mathcal{M}}(f;\zeta_1,\zeta_2)$ for bivariate functions for $0<\zeta_1\le 1$ and $0<\zeta_2\le 1$ as follows:

$$Lip_{\mathcal{M}}(f;\zeta_1,\zeta_2)=\left\{f:|f(t_1,t_2)-f(x_1,x_2)|\le \mathcal{M}|t_1-x_1|^{\zeta_1}|t_2-x_2|^{\zeta_2}\right\}.$$

Theorem 5.3.3 *If $f\in Lip_{\mathcal{M}}(f;\zeta_1,\zeta_2)$, then for $\zeta_1,\zeta_2\in(0,1]$*

$$\left|G_{n_1,n_2}^{\alpha,\beta}(f;x_1,x_2)-f(x_1,x_2)\right|\le \mathcal{M}\delta_{n_1}^{\zeta_1}\delta_{n_2}^{\zeta_2}.$$

Proof: For $f\in Lip_{\mathcal{M}}(f;\zeta_1,\zeta_2)$ and making use of Hölder's inequality, we have

$$\begin{aligned}&\left|G_{n_1,n_2}^{\alpha,\beta}(f;x_1,x_2)-f(x_1,x_2)\right|\\ &\le G_{n_1,n_2}^{\alpha,\beta}(|f(t_1,t_2)-f(x_1,x_2)|;x_1,x_2)\\ &\le \mathcal{M}G_{n_1}^{\alpha,\beta}\left(|t_1-x_1|^{\zeta_1};x_1,x_2\right)G_{n_2}^{\alpha,\beta}\left(|t_2-x_2|^{\zeta_2};x_1,x_2\right)\\ &\le \mathcal{M}\Big(G_{n_1}^{\alpha,\beta}\left((t_1-x_1)^2;x_1,x_2\right)\Big)^{\frac{\zeta_1}{2}}\left(G_{n_1}^{\alpha,\beta}(1;x_1,x_2)\right)^{\frac{2-\zeta_1}{2}}\\ &\times\left(G_{n_2}^{\alpha,\beta}\left((t_2-x_2)^2;x_1,x_2\right)\right)^{\frac{\zeta_2}{2}}\left(G_{n_2}^{\alpha,\beta}(1;x_1,x_2)\right)^{\frac{2-\zeta_2}{2}}\\ &\le \mathcal{M}\delta_{n_1}^{\zeta_1}\delta_{n_2}^{\zeta_2}.\end{aligned}$$

Theorem 5.3.4 *Let $f\in C^1(I_{\mu\kappa})$. Then, for any $(x_1,x_2)\in I_{\mu\kappa}$, we have the inequality*

$$\left|G_{n_1,n_2}^{\alpha,\beta}(f;x_1,x_2)-f(x_1,x_2)\right|\le \delta_{n_1}(x_1)\left\|f_{x_1}'\right\|_{C(I_{\mu\kappa})}+\delta_{n_2}(x_2)\left\|f_{x_2}'\right\|_{C(I_{\mu\kappa})}.$$

Proof: For $f \in C^1(I_{\mu\kappa})$, and $(x_1, x_2) \in I_{\mu\kappa}$, we have

$$f(t_1, t_2) - f(x_1, x_2) = \int_{x_1}^{t_1} f_u^{'}(u, t_2)\, du + \int_{x_2}^{t_2} f_v^{'}(x_1, v)\, dv \quad \text{for } (t_1, t_2) \in I_{\mu\kappa}.$$

Applying $G_{n_1,n_2}^{\alpha,\beta}$ to both sides, we obtain

$$G_{n_1,n_2}^{\alpha,\beta}(f; x_1, x_2) - f(x_1, x_2) \leq G_{n_1,n_2}^{\alpha,\beta}\left(\int_{x_1}^{t_1} f_u^{'}(u, t_2)\, du; x_1, x_2\right) + G_{n_1,n_2}^{\alpha,\beta}\left(\int_{x_2}^{t_2} f_v^{'}(x_1, v)\, dv\right).$$

Now, using the sup norm on $I_{\mu\kappa}$, we get

$$\left|\int_{x_1}^{t_1} f_u^{'}(u, t_2)\, du\right| \leq |t_1 - x_1| \left\|f_{x_1}^{'}\right\|_{C(I_{\mu\kappa})}$$

and

$$\left|\int_{x_2}^{t_2} f_v^{'}(x_1, v)\, dv\right| \leq |t_2 - x_2| \left\|f_{x_2}^{'}\right\|_{C(I_{\mu\kappa})}.$$

By using these inequalities, we have

$$\begin{aligned}\left|G_{n_1,n_2}^{\alpha,\beta}(f; x_1, x_2) - f(x_1, x_2)\right| &\leq G_{n_1,n_2}^{\alpha,\beta}\left(\left|\int_{x_1}^{t_1} f_u^{'}(u, t_2) du\right|; x_1, x_2\right) + G_{n_1,n_2}^{\alpha,\beta}\left(\left|\int_{x_2}^{t_2} f_v^{'}(x_1, v) dv\right|; x_1, x_2\right) \\ &\leq G_{n_1,n_2}^{\alpha,\beta}(|t_1 - x_1|; x_1, x_2) \left\|f_{x_1}^{'}\right\|_{C(I_{\mu\kappa})} + G_{n_1,n_2}^{\alpha,\beta}(|t_2 - x_2|; x_1, x_2) \left\|f_{x_2}^{'}\right\|_{C(I_{\mu\kappa})}.\end{aligned}$$

Using Cauchy Schwarz inequality, we get required result.

Next result is a Voronovoskaya type estimate for the operators (5.3).

Theorem 5.3.5 *For a function f, which is differentiable two times on the interval I, we have*

$$\begin{aligned}\lim_{n\to\infty} n\left(G_{n,n}^{\alpha,\beta}(f; x_1, x_2) - f(x_1, x_2)\right) =& \frac{\alpha}{(1 + e\beta)}\left(f_{x_1}(x_1, x_2) + f_{x_2}(x_1, x_2)\right) \\ &+ x_1 f_{x_1x_1}(x_1, x_2) + x_2 f_{x_2x_2}(x_1, x_2).\end{aligned}$$

Proof: Let $(x_1, x_2) \in I$ be arbitrary. Then by Taylor's theorem we have

$$\begin{aligned}f(t_1, t_2) =& f(x_1, x_2) + f_{x_1}(x_1, x_2)(t_1 - x_1) + f_{x_2}(x_1, x_2)(t_2 - x_2) \\ &+ \frac{1}{2}\{f_{x_1x_1}(x_1, x_2)(t_1 - x_1)^2 + 2f_{x_1x_2}(x_1, x_2)(t_1 - x_1)(t_2 - x_2) \\ &+ f_{x_2x_2}(x_1, x_2)(t_2 - x_2)^2\} + \psi(t_1, t_2; x_1, x_2)\sqrt{(t_1 - x_1)^4 + (t_2 - x_2)^4},\end{aligned} \tag{5.4}$$

where $\psi(t_1, t_2; x_1, x_2) \in C(I_{\mu\kappa})$ and $\psi(t_1, t_2; x_1, x_2) \to 0$ as $(t_1, t_2) \to (x_1, x_2)$.
Applying $G_{n,n}^{\alpha,\beta}(f; x_1, x_2)$ on both sides of (5.4), we get

$$\begin{aligned} G_{n,n}^{\alpha,\beta}(f; x_1, x_2) =& f(x_1, x_2) + f_{x_1}(x_1, x_2) G_n^{\alpha,\beta}((t_1 - x_1); x_1) + f_{x_2}(x_1, x_2) G_n^{\alpha,\beta}((t_2 - x_2); x_1) \\ &+ \frac{1}{2}\{f_{x_1x_1}(x_1, x_2) G_n^{\alpha,\beta}((t_1 - x_1)^2; x_1) + f_{x_2x_2}(x_1, x_2) G_n^{\alpha,\beta}((t_2 - x_2)^2; x_2) \\ &+ 2 f_{x_1x_2}(x_1, x_2) G_{n,n}^{\alpha,\beta}((t_1 - x_1)(t_2 - x_2); x_1, x_2)\} \\ &+ G_{n,n}^{\alpha,\beta}\left(\psi(t_1, t_2; x_1, x_2)\sqrt{(t_1 - x_1)^4 + (t_2 - x_2)^4}; x_1, x_2\right). \end{aligned} \tag{5.5}$$

Now, using Hölder's Inequality, we have

$$\begin{aligned} &\left|G_{n,n}^{\alpha,\beta}\left(\psi(t_1, t_2; x_1, x_2)\sqrt{(t_1 - x_1)^4 + (t_2 - x_2)^4}; x_1, x_2\right)\right| \\ &\leq \left\{G_{n,n}^{\alpha,\beta}(\psi^2(t_1, t_2; x_1, x_2); x_1, x_2)\right\}^{1/2}\left\{G_{n,n}^{\alpha,\beta}((t_1 - x_1)^4 + (t_2 - x_2)^4); x_1, x_2)\right\}^{1/2} \\ &\leq \left\{G_{n,n}^{\alpha,\beta}(\psi^2(t_1, t_2; x_1, x_2); x_1, x_2)\right\}^{1/2}\left\{G_n^{\alpha,\beta}((t_1 - x_1)^4; x_1) + G_n^{\alpha,\beta}((t_2 - x_2)^4; x_2)\right\}^{1/2}. \end{aligned}$$

In view of Theorem 5.3.1, $G_{n,n}^{\alpha,\beta}(\psi^2(t_1, t_2; x_1, x_2); x_1, x_2) \to 0$ as $n \to \infty$ uniformly on I and since $G_n^{\alpha,\beta}((t_1 - x_1)^4; x_1) = O\left(\frac{1}{n^2}\right)$, $G_n^{\alpha,\beta}((t_2 - x_2)^4; x_1) = O\left(\frac{1}{n^2}\right)$, therefore we have

$$\lim_{n\to\infty} n G_{n_1,n_2}^{\alpha,\beta}\left(\psi(t_1, t_2; x_1, x_2)\sqrt{(t_1 - x_1)^4 + (t_2 - x_2)^4}; x_1, x_2\right) = 0$$

uniformly on I. Also using Remark 5.2.2

$$\begin{aligned} &\lim_{n\to\infty} n G_n^{\alpha,\beta}((t_1 - x_1); x_1) = \frac{\alpha}{(1 + e\beta)}, \quad \lim_{n_2\to\infty} n(G_n^{\alpha,\beta}(t_2 - x_2); x_2) = \frac{\alpha}{(1 + e\beta)}, \\ &\lim_{n\to\infty} n(G_n^{\alpha,\beta}(t_1 - x_1)^2; x_1) = x_1, \quad \lim_{n\to\infty} n(G_n^{\alpha,\beta}(t_2 - x_2)^2; x_2) = x_2 \end{aligned}$$

and

$$\lim_{n\to\infty} n G_n^{\alpha,\beta}((t_1 - x_1); x_1) G_n^{\alpha,\beta}((t_2 - x_2); x_2) = 0.$$

Above estimate together with (5.5) gives the desired result.

In the next result, we use Peetre's K- functional defined in subsection 1.1.9 to determine the rate of convergence of bivariate operators (5.3).

Theorem 5.3.6 *For $f \in C(I_{\mu\kappa})$, the following inequality*

$$\left|\tilde{G}_{n_1,n_2}^{\alpha,\beta}(g; x_1, x_2) - g(x_1, x_2)\right| \leq 4K(f; \sigma_{n_1,n_2}(x_1, x_2)) + \varpi_2\left(f; \sqrt{\rho_{n_1,n_2}(x_1, x_2)}\right),$$

where $\rho_{n_1,n_2}(x_1, x_2) = \left(\frac{\alpha}{n_1(1+e\beta)}\right)^2 + \left(\frac{\alpha}{n_2(1+e\beta)}\right)^2$,
and $\sigma_{n_1,n_2}(x_1, x_2) = \delta_{n_1}^2(x_1) + \delta_{n_2}^2(x_2) + \rho_{n_1,n_2}(x_1, x_2)$ *holds true.*

Proof: Before initiating the proof, we define the following modified operators:

$$\tilde{G}^{\alpha,\beta}_{n_1,n_2}(g;x_1,x_2)=G^{\alpha,\beta}_{n_1,n_2}(g;x_1,x_2)-g\left(x_1+\frac{\alpha}{n_1(1+e\beta)},x_2+\frac{\alpha}{n_2(1+e\beta)}\right)+g(x_1,x_2). \quad (5.6)$$

Then using Remark (5.2.2), we have

$$\tilde{G}^{\alpha,\beta}_{n_1,n_2}((t_1-x_1);x_1,x_2)=0,\tilde{G}^{\alpha,\beta}_{n_1,n_2}((t_2-x_2);x_1,x_2)=0. \quad (5.7)$$

Let $h\in C^2(I_{\mu\kappa})$, using Taylor's theorem, we can write

$$\begin{aligned} h(t_1,t_2)-h(x_1,x_2)=&h(t_1,x_2)-h(x_1,x_2)+h(t_1,t_2)-h(t_1,x_2)\\ =&\frac{\partial h(x_1,x_2)}{\partial x_1}(t_1-x_1)+\int_{x_1}^{t_1}(t_1-u)\frac{\partial^2 h(u,x_2)}{\partial u^2}du\\ &+\frac{\partial h(x_1,x_2)}{\partial x_2}(t_2-x_2)+\int_{x_2}^{t_2}(t_2-v)\frac{\partial^2 h(x_1,v)}{\partial v^2}dv. \end{aligned} \quad (5.8)$$

Applying operators $\tilde{G}^{\alpha,\beta}_{n_1,n_2}$ on both sides of (5.8) and using (5.7), we get

$$\begin{aligned} \tilde{G}^{\alpha,\beta}_{n_1,n_2}(h;x_1,x_2)-h(x_1,x_2)=&\tilde{G}^{\alpha,\beta}_{n_1,n_2}\left(\int_{x_1}^{t_1}(t_1-u)\frac{\partial^2 h(u,x_2)}{\partial u^2}du;x_1,x_2\right)\\ &+\tilde{G}^{\alpha,\beta}_{n_1,n_2}(\int_{x_2}^{t_2}(t_2-v)\frac{\partial^2 h(x_1,v)}{\partial v^2}dv;x_1,x_2). \end{aligned}$$

Hence

$$\begin{aligned} \left|\tilde{G}^{\alpha,\beta}_{n_1,n_2}(h;x_1,x_2)-h(x_1,x_2)\right| \le& G^{\alpha,\beta}_{n_1,n_2}\left(\int_{x_1}^{t_1}|t_1-u|\left|\frac{\partial^2 h(u,x_2)}{\partial u^2}\right|du;x_1,x_2\right)\\ &+\int_{x_1}^{x_1+\frac{\alpha}{n_1(1+e\beta)}}\left|x_1+\frac{\alpha}{n_1(1+e\beta)}-u\right|\left|\frac{\partial^2 h(u,x_2)}{\partial u^2}\right|du\\ &+G^{\alpha,\beta}_{n_1,n_2}(\int_{x_2}^{t_2}|t_2-v|\left|\frac{\partial^2 h(x_1,v)}{\partial v^2}\right|dv;x_1,x_2)\\ &+\int_{x_2}^{x_2+\frac{\alpha}{n_2(1+e\beta)}}\left|x_2+\frac{\alpha}{n_2(1+e\beta)}-v\right|\left|\frac{\partial^2 h(x_1,v)}{\partial v^2}\right|dv\\ \le&\left\{G^{\alpha,\beta}_{n_1,n_2}((t_1-x_1)^2;x_1,x_2)+\left(\frac{\alpha}{n_1(1+e\beta)}\right)^2\right\}\|h\|_{C^2(I_{\mu\kappa})}\\ &+\left\{G^{\alpha,\beta}_{n_1,n_2}((t_2-x_2)^2;x_1,x_2)+\left(\frac{\alpha}{n_2(1+e\beta)}\right)^2\right\}\|h\|_{C^2(I_{\mu\kappa})}\\ \le&\left\{\delta^2_{n_1}(x_1)+\delta^2_{n_2}(x_2)+\left(\frac{\alpha}{n_1(1+e\beta)}\right)^2+\left(\frac{\alpha}{n_2(1+e\beta)}\right)^2\right\}\|h\|_{C^2(I_{\mu\kappa})}. \end{aligned}$$

Also

$$\left|\tilde{G}^{\alpha,\beta}_{n_1,n_2}(g;x_1,x_2)\right|\le 3\|g\|_{C(I_{\mu\kappa})}. \quad (5.9)$$

In view of (5.9), we can write

$$\begin{aligned}\left|\tilde{G}^{\alpha,\beta}_{n_1,n_2}(g;x_1,x_2)-g(x_1,x_2)\right| &\le \left|\tilde{G}^{\alpha,\beta}_{n_1,n_2}(g-h;x_1,x_2)\right| \\ &+\left|\tilde{G}^{\alpha,\beta}_{n_1,n_2}(h;x_1,x_2)-h(x_1,x_2)\right|+|h(x_1,x_2)-g(x_1,x_2)| \\ &+\left|g\left(x_1+\frac{\alpha}{n_1(1+e\beta)},x_2+\frac{\alpha}{n_2(1+e\beta)}\right)-g(x_1,x_2)\right| \\ &\le \left(4\|g-h\|_{C(I_{\mu\kappa})}+\sigma_{n_1,n_2}(x_1,x_2)\|h\|_{C^2(I_{\mu\kappa})}\right) \\ &+\varpi_2\left(f;\sqrt{\rho_{n_1,n_2}(x_1,x_2)}\right).\end{aligned}$$

Taking the infimum on the right hand side over all $h \in C^2(I_{\mu\kappa})$, we get

$$\left|\tilde{G}^{\alpha,\beta}_{n_1,n_2}(g;x_1,x_2)-g(x_1,x_2)\right| \le 4K(f;\sigma_{n_1,n_2}(x_1,x_2))+\varpi_2\left(f;\sqrt{\rho_{n_1,n_2}(x_1,x_2)}\right),$$

which completes the proof.

5.4 Associated GBS Operators

Let $A = I \times J$ where I and J be compact intervals. For any $f : A \to R$ and any$(t_1,t_2)\,(x_1,x_2) \in A$, let $\triangle_{(t_1,t_2)}f(x_1,x_2)$ be the bivariate mixed difference operators defined as:

$$\triangle_{(t_1,t_2)}f\,(x_1,x_2) = f(t_1,t_2) - f(t_1,x_2) - f(x_1,t_2) + f\,(x_1,x_2)\,.$$

The function $f : A \to R$ is $B-$bounded on D if there exists $K > 0$ such that $\left|\triangle_{(t_1,t_2)}f\,(x_1,x_2)\right| \le K$ for any $(t_1,t_2),(x_1,x_2) \in A$. We denote by $B_b(A)$, the space of all $B-$bounded functions on A equipped with the norm:

$$\|f\|_B = \sup_{(x_1,x_2),(t_1,t_2)\in A}\left|\triangle_{(t_1,t_2)}f\,(x_1,x_2)\right|.$$

Further let $C_b\,(A)$ be the subspace consisting of the functions from $B_b\,(A)$ and is defined by

$$C_b\,(A) = \left\{f|\lim_{(t_1,t_2)\to(x_1,x_2)}\triangle_{(t_1,t_2)}f\,(x_1,x_2) = 0\forall\,(x_1,x_2)\in A\right\}.$$

Here, for a compact subset A, every $B-$continuous function is a B-bounded function.
We denote the space of all $B-$differentiable functions on A by

$$D_b\,(A) = \left\{f|f : A \to R, and \lim_{(t_1,t_2)\to(x_1,x_2)}\frac{\triangle_{(t_1,t_2)}f\,(x_1,x_2)}{(t_1-x_1)\,(t_2-x_2)} = D_B f\,(x_1,x_2) < \infty\forall\,(x_1,x_2)\in A\right\}.$$

The mixed modulus of continuity of $f \in B_b\left(I_{\mu\kappa}\right)$, where $I_{\mu\kappa} = [0,\mu]\times[0,\kappa]$, is the compact subset of A, is the function $\omega_B : [0,\infty)\times[0,\infty)\to\mathbb{R}$, defined by

$$\omega_B\,(f;\delta_1,\delta_2) := \sup\{|\Delta f\,[(t,s);(x_1,x_2)]| : |x_1-t_1|<\delta_1, |x_2-t_2|<\delta_2\},$$

for any $(t_1, t_2), (x_1, x_2) \in A$.
We define the GBS of the operators $G^{\alpha,\beta}_{n_1,n_2}$ given by (5.3), for any $f \in C_b\left(I_{\mu\kappa}\right)$ and $n, m \in \mathbb{N}$, by

$$\widehat{G}^{\alpha,\beta}_{n_1,n_2}\left(f(t_1,t_2); x_1, x_2\right) = G^{\alpha,\beta}_{n_1,n_2}\left(f(t_1,x_2) + f(x_1,t_2) - f(t_1,t_2); x_1, x_2\right), \tag{5.10}$$

that is

$$\widehat{G}^{\alpha,\beta}_{n_1,n_2}(f; x_1, x_2) = e^{-(n_1x_1+n_2x_2)}\left(\frac{1+e\beta}{2}\right)^{2\alpha}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{g^{\alpha}_{k_1}(n_1x_1;\beta)}{k_1!}\frac{g^{\alpha}_{k_2}(n_2x_2;\beta)}{k_2!}\left(f\left(\frac{k_1}{n_1},x_2\right)\right.$$
$$\left.+f\left(x_1,\frac{k_2}{n_2}\right) - f\left(\frac{k_1}{n_1},\frac{k_2}{n_2}\right)\right).$$

Theorem 5.4.1 *For every $f \in C_b(I_{\mu\kappa})$, at each point $(x_1, x_2) \in I_{\mu\kappa}$, the operators* (5.10) *verifies the following inequality*

$$\left|\widehat{G}^{\alpha,\beta}_{n_1,n_2}(f; x_1, x_2) - f(x_1, x_2)\right| \leq 4\omega_B\left(f; \delta_{n_1}, \delta_{n_2}\right).$$

Proof: By using the property of $\omega_B\left(f; \delta_{n_1}, \delta_{n_2}\right)$, we have

$$\left|\Delta_{(t_1,t_2)}f(x_1,x_2)\right| \leq \omega_B\left(f; |t_1-x_1|, |t_2-x_2|\right) \leq \left(1 + \frac{|t_1-x_1|}{\delta_{n_1}}\right)\left(1 + \frac{|t_2-x_2|}{\delta_{n_2}}\right)\omega_B\left(f; \delta_{n_1}, \delta_{n_2}\right).$$

Now

$$\begin{aligned}\left|\widehat{G}^{\alpha,\beta}_{n_1,n_2}(f; x_1, x_2) - f(x_1, x_2)\right| &\leq G^{\alpha,\beta}_{n_1,n_2}\left(\left|\Delta_{(t_1,t_2)}f(x_1,x_2)\right|; x_1, x_2\right)\\
&\leq \left(G^{\alpha,\beta}_{n_1,n_2}(1; x_1, x_2) + \frac{1}{\delta_{n_1}}\left(G^{\alpha,\beta}_{n_1,n_2}\left((t_1-x_1)^2; x_1, x_2\right)\right)^{1/2}\right.\\
&+ \frac{1}{\delta_{n_2}}\left(G^{\alpha,\beta}_{n_1,n_2}\left((t_2-x_2)^2; x_1, x_2\right)\right)^{1/2}\\
&+ \frac{1}{\delta_{n_1}}\left(G^{\alpha,\beta}_{n_1,n_2}\left((t_1-x_1)^2; x_1, x_2\right)\right)^{1/2}\\
&\left.\frac{1}{\delta_{n_2}}\left(G^{\alpha,\beta}_{n_1,n_2}\left((t_2-x_2)^2; x_1, x_2\right)\right)^{1/2}\right)\omega_B\left(f; \delta_{n_1}, \delta_{n_2}\right)\\
&\leq 4\omega_B\left(f; \delta_{n_1}, \delta_{n_2}\right).\end{aligned}$$

Next, let us define the Lipschitz class for $B-$continuous functions. For $f \in C_b\left(I_{\mu\kappa}\right)$ the Lipschitz class $Lip_{\mathcal{M}}(\zeta_1, \zeta_2)$ with $\zeta_1, \zeta_2 \in (0, 1]$ is defined by

$$Lip_M(\zeta_1,\zeta_2) = \left\{f \in C_b\left(I_{\mu\kappa}\right) : \left|\Delta f_{(t_1,t_2)}(x_1,x_2)\right| \leq \mathcal{M}|t_1-x_1|^{\zeta_1}|t_2-x_2|^{\zeta_2}, \text{for } (t_1,t_2), (x_1,x_2) \in I_{\mu\kappa}\right\}.$$

In our next result, we determine the order of approximation for the operators $\widehat{G}^{\alpha,\beta}_{n_1,n_2}$ by means of the class $Lip_{\mathcal{M}}(\zeta_1, \zeta_2)$.

Theorem 5.4.2 *For $f \in Lip_M(\zeta_1, \zeta_2)$, we have*

$$|\widehat{G}^{\alpha,\beta}_{n_1,n_2}(f; x_1, x_2) - f(x_1, x_2)| \leq \mathcal{M}(\delta_{n_1}(x_1))^{\zeta}_1(\delta_{n_2}(x_2))^{\zeta}_2,$$

for $\mathcal{M} > 0$, $\zeta_1, \zeta_2 \in (0, 1]$.

Theorem 5.4.3 *If $f \in D_b\left(I_{\mu\kappa}\right)$ and $D_B f \in B\left(I_{\mu\kappa}\right)$, then for each $(x_1, x_2) \in I_{\mu\kappa}$, we get*

$$\left|\widehat{G}^{\alpha,\beta}_{n_1,n_2}(f; x_1, x_2) - f(x_1, x_2)\right| \leq \frac{M}{n_1^{1/2} n_2^{1/2}} \left(\|D_B f\|_\infty + \omega_B\left(D_B f; n_1^{-1/2}, n_2^{-1/2}\right)\right).$$

Proof: Since $f \in D_b\left(I_{\mu\kappa}\right)$, we have the identity

$$\triangle f\left[(t_1, t_2); (x_1, x_2)\right] = (t_1 - x_1)(t_2 - x_2) D_B(\varepsilon, n), \text{ with } x_1 < \varepsilon < t_1; x_2 < n < t_2.$$

Eventually,

$$D_B f(\varepsilon, n) = \triangle D_B f(\varepsilon, n) + D_B f(\varepsilon, x_2) + D_B f(x_1, n) - D_B f(x_1, x_2).$$

Since $D_B f \in B\left(I_{\mu\kappa}\right)$, by above relation, we can write

$$\begin{aligned}
\left|G^{\alpha,\beta}_{n_1,n_2}\left(\triangle f\left[(t_1, t_2); (x_1, x_2)\right]; x_1, x_2\right)\right| &= \left|G^{\alpha,\beta}_{n_1,n_2}\left((t_1 - x_1)(t_2 - x_2) D_B(\varepsilon, n); x_1, x_2\right)\right| \\
&\leq G^{\alpha,\beta}_{n_1,n_2}\left(|t_1 - x_1|\,|t_2 - x_2| \triangle D_B f(\varepsilon, n); x_1, x_2\right) \\
&+ G^{\alpha,\beta}_{n_1,n_2}\left(|t_1 - x_1|\,|t_2 - x_2| \left(|D_B f(\varepsilon, x_2)|\right.\right. \\
&+ \left.\left. |D_B f(x_1, n)| + |D_B f(x_1, x_2)|\right); x_1, x_2\right) \\
&\leq G^{\alpha,\beta}_{n_1,n_2}\left(|t_1 - x_1|\,|t_2 - x_2|\, \omega_B\left(D_B f; |\varepsilon - x_1|, |n - x_2|\right); x_1, x_2\right) \\
&+ 3\|D_B f\|_\infty G^{\alpha,\beta}_{n_1,n_2}\left(|t_1 - x_1|\,|t_2 - x_2|; x_1, x_2\right).
\end{aligned}$$

Using above inequality along with linearity of $G^{\alpha,\beta}_{n_1,n_2}$ and Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\left|\widehat{G}^{\alpha,\beta}_{n_1,n_2}(f; x_1, x_2) - f(x_1, x_2)\right| &= \left|G^{\alpha,\beta}_{n_1,n_2}\left(f\left[(t_1, t_2); (x_1, x_2)\right]; x_1, x_2\right)\right| \\
&\leq 3\|D_B f\|_\infty \sqrt{G^{\alpha,\beta}_{n_1,n_2}\left((t_1 - x_1)^2 (t_2 - x_2)^2; x_1, x_2\right)} \\
&+ \left(G^{\alpha,\beta}_{n_1,n_2}\left(|t_1 - x_1|\,|t_2 - x_2|; x_1, x_2\right)\right. \\
&+ \delta_{n_1}^{-1} G^{\alpha,\beta}_{n_1,n_2}\left((t_1 - x_1)^2 |t_2 - x_2|; x_1, x_2\right) \\
&+ \delta_{n_2}^{-1} G^{\alpha,\beta}_{n_1,n_2}\left(|t_1 - x_1| (t_2 - x_2)^2; x_1, x_2\right) \\
&+ \delta_{n_1}^{-1} \delta_{n_2}^{-1} G^{\alpha,\beta}_{n_1,n_2}\left((t_1 - x_1)^2 (t_2 - x_2)^2; x_1, x_2\right) \omega_B\left(D_B f; \delta_{n_1}, \delta_{n_2}\right) \\
&\leq 3\|D_B f\|_\infty \sqrt{G^{\alpha,\beta}_{n_1,n_2}\left((t_1 - x_1)^2 (t_2 - x_2)^2; x_1, x_2\right)} \\
&+ \left(\sqrt{G^{\alpha,\beta}_{n_1,n_2}\left((t_1 - x_1)^2 (t_2 - x_2)^2; x_1, x_2\right)}\right. \\
&+ \delta_{n_1}^{-1} \sqrt{G^{\alpha,\beta}_{n_1,n_2}\left((t_1 - x_1)^4 (t_2 - x_2)^2; x_1, x_2\right)} \\
&+ \delta_{n_2}^{-1} \sqrt{G^{\alpha,\beta}_{n_1,n_2}\left((t_1 - x_1)^2 (t_2 - x_2)^4; x_1, x_2\right)} \\
&\left. + \delta_{n_1}^{-1} \delta_{n_2}^{-1} G^{\alpha,\beta}_{n_1,n_2}\left((t_1 - x_1)^2 (t_2 - x_2)^2; x_1, x_2\right)\right) \omega_B\left(D_B f; \delta_{n_1}, \delta_{n_2}\right).
\end{aligned} \tag{5.11}$$

In view of Remark 5.2.2, for $(t_1, t_2) \in I_{\mu\kappa}, (x_1, x_2) \in I_{\mu\kappa}$ and $i, j = 1, 2$

$$\begin{aligned}
G^{\alpha,\beta}_{n_1,n_2}\left((t_1 - x_1)^{2i} (t_2 - x_2)^{2j}; x_1, x_2\right) &= G^{\alpha,\beta}_{n_1,n_2}\left((t_1 - x_1)^{2i}; x_1, x_2\right) G^{\alpha,\beta}_{n_1,n_2}\left((t_2 - x_2)^{2j}; x_1, x_2\right) \\
&\leq \frac{M_1}{n_1^i} \frac{M_2}{n_2^j},
\end{aligned} \tag{5.12}$$

for some constant $M_1, M_2 > 0$.

Choosing $\delta_{n_1} = n_1^{-1/2}$, and $\delta_{n_2} = n_2^{-1/2}$ and combining (5.11) and (5.12), we get the desired result.

Example 5.4.4 *Figure 5.1 illustrates the convergence of operators* $G_{n_1,n_2}^{\alpha,\beta}$ *to* $f(x_1, x_2) = (x_1 - 1)^2 - (x_2 - 1)^2$ *for* $\alpha = \beta = 2$

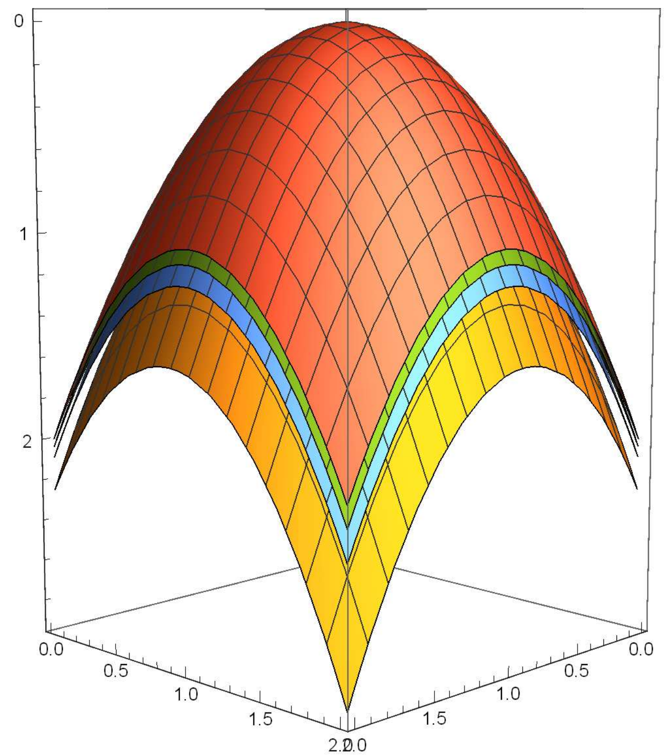

Figure 5.1: The Convergence of operators $G_{n_1,n_2}^{\alpha,\beta}$ to the function $f(x_1, x_2)$ (Red f, Green $G_{50,50}^{2,2}$, Blue $G_{20,20}^{2,2}$, Yellow $G_{5,5}^{2,2}$)

Example 5.4.5 *In Table 5.1, we estimate the absolute error* $E_{n_1,n_2}^{\alpha,\beta}(f; x_1, x_2) = \left|G_{n_1,n_2}^{\alpha,\beta}(f; x_1, x_2) - f(x_1, x_2)\right|$ *for functions* $f_1(x_1, x_2) = (x_2 - 2)^2 - (x_1 - 2)^2$ *and* $f_2(x_1, x_2) = x_1 - 2x_1x_2 + 1$. *Further, the convergence of the operators* $G_{n_1,n_2}^{\alpha,\beta}$ *to the function* $f_1(x_1, x_2)$ *is illustrated in Figure 5.2. Both table and figure show a very interesting fact that the convergence of operators* $G_{n_1,n_2}^{\alpha,\beta}$ *is faster when* $\alpha < \beta$ *and slower when* $\beta < \alpha$.

Table 5.1: Error estimation of operators $G^{\alpha,\beta}_{n_1,n_2}$ to the functions $f_1(x_1,x_2)$ and $f_2(x_1,x_2)$ (for cases $\alpha > \beta, \beta > \alpha$)

(x_1,x_2)	$E^{5,4}_{15,15}$	$E^{4,5}_{15,15}$
(1,1)	0.0858002	0.0554948
(2,2)	0.198099	0.128597
(3, 3)	0.310397	0.2017
(4,4)	0.422695	0.274802
(5,5)	0.534994	0.347904

(x_1,x_2)	$E^{6,2}_{5,5}$	$E^{2,6}_{5,5}$
(0,4)	2.29148	0.984868
(0.5, 3.5)	1.71861	0.738651
(1, 3)	1.14574	0.492434
(1.5, 2.5)	0.57287	0.246217
(2, 2)	0	0

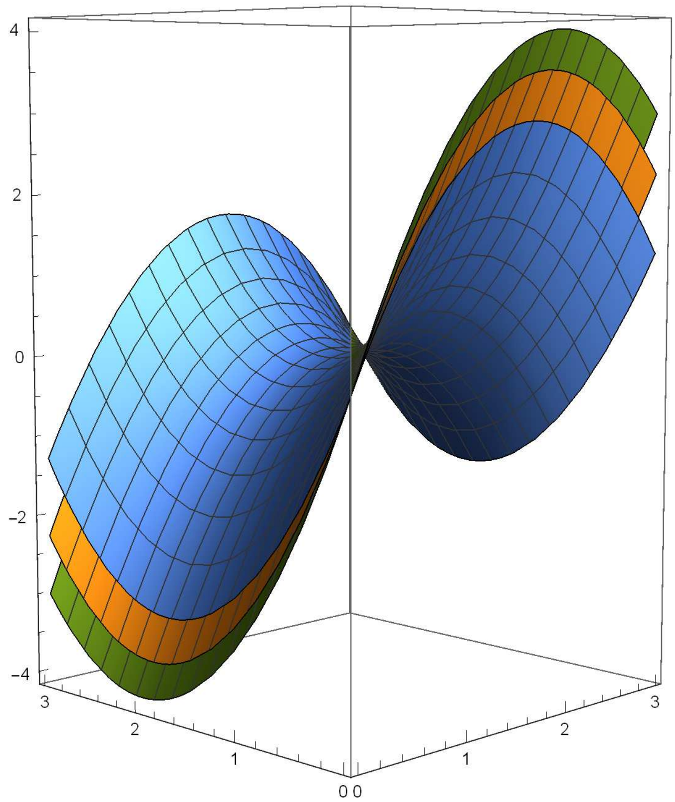

Figure 5.2: Convergence of operators $G^{\alpha,\beta}_{n_1,n_2}$ to the function $f_1(x_1,x_2)$ (Green f_1, yellow $G^{2,6}_{5,5}$, ($\beta > \alpha$), Blue $G^{6,2}_{5,5}$, ($\alpha > \beta$))

Example 5.4.6 *Define* $\widehat{E}^{\alpha,\beta}_{n_1,n_2} = \left|\widehat{G}^{\alpha,\beta}_{n_1,n_2}(f;x_1,x_2) - f(x_1,x_2)\right|$. *In Table 5.2, we estimate the error between the operators* $G^{\alpha,\beta}_{n_1,n_2}$ *and its corresponding GBS* $\widehat{G}^{\alpha,\beta}_{n_1,n_2}$. *We observe that GBS operators converge to the function parallely and the convergence is faster as well.*

Table 5.2: Comparison between bivariate and its corresponding GBS for the function $f_2(x_1, x_2)$

(a) Considering equal values of n_1 and n_2

(x_1, x_2)	$E_{1,1}^{1,10}$	$\widehat{E}_{1,1}^{1,10}$
(0.4,0.4)	0.0238076	0.00251803
(0.8,0.8)	0.0805798	0.00251803
(1.2,1.2)	0.137352	0.00251803
(1.6,1.6)	0.194124	0.00251803
(2,2)	0.250896	0.00251803

(b) Considering different values of n_1 and n_2

(x_1, x_2)	$E_{8,15}^{1,10}$	$\widehat{E}_{8,15}^{1,10}$
(2,2)	0.022789	0.0000209836
(4,4)	0.0499923	0.0000209836
(6,6)	0.0771957	0.0000209836
(8,8)	0.104399	0.0000209836
(10,10)	0.131602	0.0000209836

Conclusion and Future scope

Conclusion

The aim of this chapter is to present a concluding remarks to our thesis and illustrate some of the prospects that define our current and future endeavours in scientific research.

This thesis is mainly a study of convergence estimates of various approximation operators. The introductory chapter consists of definitions and literature survey of concepts used throughout this thesis. In the second chapter, we discuss approximation operators of exponential type. The first section of this chapter presents the study of convergence estimates of Bézier variant of Ismail-May operators. Further we also propose a two variable generalization of these operators. In the second section, we present a modification of Ismail-May exponential operators which preserve exponential functions.

Chapter three presents a conceptual theoretical framework based on the modification of certain Gamma type operators that preserve the test functions $t^{\vartheta},\ \vartheta = \{0\} \bigcup \mathbb{N}$. We deduce numerically as well as graphically that the modified operators approximate best while they preserve the test function t^3. We further examine the rate of convergence of the modified operators in terms of first and second order modulus of continuity and in the sense of Peetre's K-functional. Finally, we conclude with a theorem establishing the degree of approximation for functions of bounded variation.

The study of chapter four is concerned with generalization of Bernstein operators. In section one, we propose a Pólya distribution-based generalization of λ-Bernstein operators. We establish some fundamental results for convergence as well as order of approximation of the proposed operators. We present theoretical result and graph to demonstrate the proposed operator's intriguing ability to interpolate the interval's end point. In order to illustrate the convergence of proposed operators as well as the impact of changing the parameter "μ", we provide a variety of graphs as our paper's conclusion.

Usta provided a modification of Bernstein operators in 2020 that was suited for approximation on $(0, 1)$. We define generalized Bernstein operators with shifted knots in this study. Shifted knots have the benefit of allowing approximation on interval $(0, 1)$ and its subinterval. It also increases the flexibility of operators for approximation. Certain theorems are derived to verify the convergence of our newly constructed operators. In addition, we provide weighted approximation theorem, Voronovskaja and Grüss Voronovskaja type theorems to demonstrate asymptotic behaviour. Graphs and tables validate the convergence and show the approximation error.

In chapter five, we propose the bivariate generalization of operators involving Apostol-Genocchi polynomial. We obtain the rate of convergence in terms of partial and total modulus of continuity and order of approximation by means of a Lipschitz type function and the Peetre's K-functional. Also we propose a generalization called "generalized boolean sum (GBS)" of these bivariate operators to determine the order of approximation for Bögel continuous functions. Employing Mathematica Software, we show a few graphical examples to demonstrate the rate of convergence for the bivariate operators. It gets known through those graphs that for some particular functions the bivariate operators converges better when $\alpha < \beta$. After analyzing and comparing the error of approximation of the bivariate operators and the associated GBS operators we conclude that the GBS operators converges parallel and faster to the function.

Academic future plans

I plan to carry out more study in the field of approximation by positive linear operators and hopefully share my results with the mathematical community. Two decades ago it was observed that if we modify the original operators, we can have a better approximation. Another interesting problem in the theory of approximation is to present parametric generalizations of existing operators which enables us to approximate a wider space of functions also it increases the flexibility of operators for approximation. This thesis includes some modifications and parametric generalization of the existing operators. In addition to work on existing operators in this area, I shall strive to study and contribute towards developing new operators.

In this thesis, we have worked on the modification of operators which preserves exponential function. We have given several fundamental theorems to show uniform convergence of these modified operators and analyzed their aymptotic behavior through the Voronovskaya-type theorem. In future, I intend to investigate their iterates, invariant measures, Kantorovich modifications, eigenstructure, and global smoothness preservation properties.

In this thesis, we have also studied the modifications of certain exponential operators. The exponential-type operators were introduced four decades ago and since then no new exponential-type operators were introduced although several generalizations of existing exponential-type operators were proposed and studied. In [102], it was established that corresponding to each polynomial $p(x)$, a unique approximation operator can be obtained which satisfies the differential equation (1.11) and normalization condition (1.12). Operators corresponding to constant, linear, quadratic and cubic polynomials were obtained using the method of bilateral Laplace transform. In future, I shall attempt to seek the methods that can determine the exponential operators associated with higher order polynomials.

Very recently, Tyliba and Wachnicki [166] extended the work of Ismail-May [102] with a more general family of operators by introducing a new parameter β. These operators were termed as semi-exponential operators and studied by Herzog [99]. The corresponding semi-exponential operators for Bernstein, Baskakov and Ismail-May operators were obtained by Abel et al. [3]. Since the area of semi exponential operators is not much explored, I intend to study the approximation properties of these operators such as the complete asymptotic expansion, behaviour of their derivatives through simultaneous approximation and other convergence estimates.

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List of Publications

1. Km. Lipi, Naokant Deo: General family of exponential operators. *Filomat*, 34(12), 4043-4061 (2020). https://doi.org/10.2298/FIL2012043L **(SCIE, Impact Factor 0.988)**

2. Km. Lipi, Naokant Deo: On modification of certain exponential type operators preserving constant and e^{-x}. *Bulletin of the Malaysian Mathematical Sciences Society*, 44(5), 3269-3284 (2021). https://doi.org/10.1007/s40840-021-01100-3 **(SCIE, Impact Factor 1.2)**

3. Km. Lipi, Naokant Deo: Approximation properties of modified Gamma operators preserving t^{ϑ}. *Annals of Functional Analysis*, 13, Article 26 (2022). https://doi.org/10.1007/s43034-022-00172-x **(SCIE, Impact Factor 1)**

4. Km. Lipi, Naokant Deo: λ-Bernstein operators based on Pólya distribution. *Numerical Functional Analysis and Optimization*, 44(6), 529-544 (2023). https://doi.org/10.1080/01630563.2023.2185896 **(SCIE, Impact Factor 1.2)**

5. Naokant Deo, Km. Lipi: Approximation by means of modified Bernstein operators with shifted knots. *The Journal of Analysis*, (2023). https://doi.org/10.1007/s41478-023-00681-5 **(SCOPUS, Impact Factor 0.8)**

6. Km. Lipi, Naokant Deo: Bivariate generalization for operators involving Appostol-Genocchi polynomial. **(Communicated)**

Papers presented in International Conferences

1. Bézier variant of certain exponential type operators; *International Conference on Advances in Multi-Disciplinary Sciences and Engineering Research* (ICAMSER-2021), Chitkara University, Himachal Pradesh. July 2-3, 2021

2. A note on Ismail-May operators preserving the exponential function; *Emerging trends in Pure and Applied Mathematics*, Tezpur University, Assam. March 12-13, 2022